Hodge Polynomials of Moduli Spaces of Stable Pairs on K3 Surfaces

Benjamin T. Bakker

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Abstract

Virtual curve counts have been defined for threefolds by integration against virtual classes on moduli spaces of stable maps (Gromov-Witten theory), ideal sheaves (Donaldson-Thomas theory), and stable pairs (Pandharipande-Thomas theory). The first two theories are proven to be equivalent for toric threefolds, and all three are conjecturally equivalent for arbitrary threefolds. One may ask whether there is such a correspondence for surfaces. In particular, the Gromov-Witten theory of K3 surfaces has recently been computed by Maulik, Pandharipande, and Thomas; it is governed by quasimodular forms and is closely related to invariants obtained from the moduli spaces of rank $r = 0$ stable pairs with $n = 1$ sections. We compute the Hodge polynomials of the moduli spaces of stable pairs for higher rank $r \geq 0$ and level $n \geq 1$, and explore the modularity properties and relationship to Gromov-Witten theory.

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My family has been a constant source of support throughout my studies, and I would like to thank my five brothers—Andrew, Joshua, Jonathan, Jacob, and Alexander—for providing welcome distractions. Last but not least, I am eternally grateful to my parents Donald and Grace, to whom this thesis is dedicated. I am who I am today because of their love and encouragement.

To my parents.

Contents

Chapter 1

Introduction

1.1 Gromov-Witten Theory

Gromov-Witten theory was originally developed to make mathematically rigorous counts of embedded curves but has since revealed a rich structure of its own. For simplicity take X to be a smooth algebraic variety over $\mathbb{C}$ and $\beta \in H_2(X, \mathbb{Z})$ a curve class. Kontsevich and Manin [KM94] constructed a Deligne-Mumford stack $\overline{M}_{g,n}(X,\beta)$ of stable maps $C \to X$ from genus g, n-pointed curves C whose image has homology class β. Following [BF97], $\overline{M}_{g,n}(X,\beta)$ carries a perfect obstruction theory and therefore has a virtual fundamental class $[M_{g,n}(X,\beta)]^{vir} \in A_*(M_{g,n}(X,\beta))$ of dimension

$$\dim[\overline{M}_{g,n}(X,\beta)]^{vir} = \int_\beta c_1(X) + (3 - \dim X)(g-1) + n \tag{1.1}$$

Let $\pi : \mathcal{C} \to \overline{M}_{g,n}(X,\beta)$ be the universal curve and $\mu : \mathcal{C} \to X$ the universal stable map. There are n sections $\sigma_i : \overline{M}_{g,n}(X,\beta) \to \mathcal{C}$ corresponding to the n marked points; the ith evaluation map is the composition

$$\mathrm{ev}_i = \mu\sigma_i : \overline{M}_{g,n}(X,\beta) \to X$$

which on the level of $\mathbb{C}$-points maps a stable map $f : C \to X$ with ith marked point $p_i \in C$ to $f(p_i)$. $\mathcal{C}$ is a flat curve over $\overline{M} = \overline{M}_{g,n}(X, \beta)$ with only nodal singularities, so the relative dualizing sheaf $\omega = \omega_{\mathcal{C}/\overline{M}}$ is a line bundle. The ψ_i classes are defined as

$$\psi_i = \mathrm{c}_1(\sigma_i^* \omega) \in A^1(\overline{M}_{g,n}(X, \beta))$$

and the Gromov-Witten invariants are all integrals of the form

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_{g,\beta}^X = \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^n \psi_i^{k_i} \cup \mathrm{ev}_i^*(\gamma_i) \tag{1.2}$$

where $k_i \in \mathbb{N}$ and $\gamma_i \in A^*(X)$. The Gromov-Witten invariant (1.2) attaches a numerical invariant to the collection of n-pointed curves in X meeting prescribed subvarieties of homology classes dual to the γ_i at the ith marked point with k_i-fold tangency. Of particular importance are threefolds X for which $c_1(X) = 0$—so called Calabi-Yau threefolds—since the moduli spaces of 0-pointed stable maps all have virtual dimension 0 by (1.1). The Gromov-Witten numbers[1]

$$N_{g,\beta} = \int_{[\overline{M}_g(X,\beta)]^{vir}} 1$$

are virtual curve counts. The reduced Gromov-Witten potential is

$$F'_{GW}(X; u, v) = \sum_{\beta \neq 0, g \geq 0} N_{g,\beta} u^{2g-2} v^\beta$$

and the reduced Gromov-Witten partition function

$$Z'_{GW}(X; u, v) = \exp F'_{GW}(X; u, v)$$

generates Gromov-Witten invariants with possibly disconnected domain curves.

[1] $\overline{M}_g(X, \beta) = \overline{M}_{g,0}(X, \beta)$ is the moduli space of unmarked stable curves.

1.2 Donaldson-Thomas Theory on Threefolds

A second method for virtually counting curves on threefolds was introduced in [Tho00] using the Hilbert scheme of 1-dimensional subschemes. Let X be a smooth 3-dimensional algebraic variety over $\mathbb{C}$, $\beta \in H_2(X, \mathbb{Z})$ a curve class, and $I_n(X, \beta)$ the Hilbert scheme of 1-dimensional subschemes Z with $[Z] = \beta$ and $\chi(\mathcal{O}_Z) = n$. The ideal sheaf $\mathcal{I}_Z$ of such a Z is reflexive, rank 1, and has trivial determinant. Conversely any such sheaf $\mathcal{I}$ injects into its double dual

$$\mathcal{I} \hookrightarrow \mathcal{I}^{\vee\vee} \cong \mathcal{O}$$

and is therefore an ideal sheaf. $I_n(X, \beta)$ may thus be thought of as the moduli space of reflexive rank 1 sheaves with trivial determinant. $I_n(X, \beta)$ has a perfect obstruction theory which for X Calabi-Yau has virtual dimension 0, and the resulting invariants

$$N'_{n,\beta} = \int_{[I_n(X,\beta)]^{vir}} 1$$

are the Donaldson-Thomas invariants[2]. We likewise form the total Donaldson-Thomas partition function

$$Z_{DT}(X; q, v) = \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{n \in \mathbb{Z}} N'_{n,\beta} q^n v^\beta$$

The subschemes parametrized by $I_n(X, \beta)$ for $\beta \neq 0$ must have a 1-dimensional component in the curve class β, but they may also include 0-dimensional components not supported on the curve. To correct for this, we define the reduced Donaldson-Thomas partition function

$$Z'_{DT}(X; q, v) = \frac{Z_{DT}(X; q, v)}{Z_{DT}(X; q)_0}$$

[2]For X not Calabi-Yau, invariants with insertions analagous to (1.2) may be defined using the Chern classes of the universal ideal sheaf in place of the ψ_i classes.

where

$$Z_{DT}(X;q)_0 = \sum_{n\in\mathbb{Z}} N'_{n,0} q^n$$

is the degree 0 partition function. It has been computed; let

$$M(q) = \prod_{n\geq 0} \frac{1}{(1-q^n)^n}$$

be the McMahon function, the generating function of 3-dimensional partitions. Then

Theorem 1.2.1. *[Li06, BF08, LP] For X a threefold*

$$Z_{DT}(X;q)_0 = M(-q)^{\chi(X)}$$

where $\chi(X)$ is the topological Euler characteristic.

The main result of [MNOP06a, MNOP06b] is that the reduced Gromow-Witten and Donaldson-Thomas partition functions are related by the change of variable $q = -e^{iu}$.

Theorem 1.2.2. *[MNOP06a, MNOP06b] For X a toric Calabi-Yau threefold*[3]

$$Z'_{GW}(X;u,v) = Z'_{DT}(X;-e^{iu},v)$$

(1.2.2) has been proven for the generating functions of primary invariants for all toric threefolds as well [MOOP]; primary invariants are of the form (1.2) with all $k_i = 0$. Thus there is a strong relationship between virtual curve counts via stable maps and curve counts via moduli of sheaves.

[3]There are no proper toric Calabi-Yau threefolds, but GW and DT invariants can still be defined by equivariant localization for local Calabi-Yau threefolds.

1.3 Stable Pairs

A third alternative to virtually counting curves on a threefold X was developed in [PT09b, PT09a, PT10] using Le Potier's theory of stable pairs[LP93, LP95]. A stable pair on X consists of a purely 1-dimensional sheaf F on X and a section $\mathcal{O} \to F$ with 0-dimensional cokernel. Given a smooth curve $C \subset X$ and a divisor D on C, we can associate a stable pair $\mathcal{O} \to \mathcal{O}_C(D)$ whose cokernel is supported on the points of D, and the moduli space $P_n(X, \beta)$ of stable pairs $\mathcal{O} \to F$ with $\chi(F) = n$ and $[\mathrm{Supp}(F)] = \beta$ may thus be thought of as a compactification of the moduli spaces of embedded n-pointed smooth curves. Pandharipande and Thomas show that $P_n(X, \beta)$ has a virtual class which for X Calabi-Yau has dimension 0; the invariants

$$N''_{n,\beta} = \int_{[P_n(X,\beta)]^{vir}} 1$$

are the Pandharipande-Thomas invariants, and they are also closely related to Gromov-Witten and Donaldson-Thomas counts. In fact, if

$$Z'_{PT} = \sum_{\beta \neq 0} \sum_{n \in \mathbb{Z}} N''_{n,\beta} q^n v^\beta$$

is the reduced Pandharipande-Thomas partition function, then

Theorem 1.3.1. *[PT09b]*

$$Z'_{DT}(X; q, v) = Z'_{PT}(X; q, v)$$

Theorem (1.3.1) was treated in the toric case by [PT09a]; it was observed in [PT09b] that the equality can be viewed as a wall-crossing formula for invariants of stability conditions on $D^b(X)$. The general case of the theorem has been treated by many authors, [Tod, ST, Bri].

1.4 The Gromov-Witten Theory of K3 Surfaces

It is natural to ask whether there is an analogous relationship between the Gromov-Witten theory of surfaces and sheaf-theoretic virtual curve counts of surfaces. In the threefold case, the relationship is most easily described in the Calabi-Yau case, so it is natural to ask the question first for K3 surfaces. The Gromov-Witten theory of K3 surfaces has been studied recently by Maulik, Pandharipande, and Thomas [MPT], and the Gromov-Witten partition functions have proved to have surprising modularity properties.

Let X be a K3 surface, β a curve class on X. The normal Gromov-Witten theory of X vanishes because the obstruction bundle has a canonical trivial quotient. Indeed, the obstruction space at a stable map $f : C \to X$ is $H^1(f^*T_X)$, but since $\omega_X \cong \mathcal{O}_X$, the canonical map

$$H^1(f^*T_X) \cong H^1(f^*\Omega^1_X) \to H^1(\omega_C) \cong \mathbb{C}$$

yields a trivial quotient Obs $\to \mathcal{O}$ of the obstruction bundle Obs. This forces the virtual class to be 0 since naively $[\overline{M}_{g,n}(X,\beta)]^{vir}$ is the Euler class of the obstruction bundle.

After modifying the obstruction theory by taking instead the kernel of Obs $\to \mathcal{O}$ to be the obstruction bundle, we obtain a reduced virtual class $[\overline{M}_{g,n}(X,\beta)]^{red}$ with virtual dimension one greater than expected:

$$\dim[\overline{M}_{g,n}(X,\beta)]^{red} = 1 + \int_\beta \mathrm{c}_1(T_X) + (3 - \dim X)(g-1) + n = g + n$$

The reduced Gromov-Witten invariants are

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle^{X,red}_{g,\beta} = \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^n \psi_i^{k_i} \cup \mathrm{ev}_i^*(\gamma_i) \qquad (1.3)$$

The main result is

Theorem 1.4.1. *[MPT] Let X be an elliptic K3 with section, s the section class and f the fiber class. Each Gromov-Witten potential*

$$F_g^X(\tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)) = \sum_{h\geq 0} \langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle_{g,s+hf}^{X,red} q^{h-1}$$

is the Fourier expansion of a quasimodular form.

In addition to the ψ_i classes, one can also define the Hodge classes $\lambda_i = c_i(\pi_*\omega)$, where ω is the relative dualizing sheaf of the universal curve $\mathcal{C} \xrightarrow{\pi} \overline{M}_{g,n}(X,\beta)$. Let

$$R_{g,\beta} = \int_{[\overline{M}_g(X,\beta)]^{red}} (-1)^g \lambda_g$$

and define the partition function

$$Z_\beta^{GW}(u) = \sum_{g\geq 0} R_{g,\beta} u^{2g-2} \tag{1.4}$$

$Z_\beta^{GW}(u)$ only depends on the genus h of β by deformation invariance, and we will also denote (1.4) by $Z_h^{GW}(u)$.

1.5 Sheaf-Counting on K3 Surfaces

Let X be a K3 surface and D a divisor class such that every divisor in D is reduced and irreducible of genus g. Let $\mathbb{P} = |D|$ be the complete linear system of D and $X\times\mathbb{P} \supset \mathcal{C}_g \to \mathbb{P}$ the universal divisor. The relative Hilbert scheme $\mathcal{C}_g^{[d]} = \mathrm{Hilb}^d(\mathcal{C}_g/\mathbb{P})$ parametrizing divisors C in the class D and subschemes Z of C of length n is the surface analog of the moduli space $P_n(X,\beta)$ of stable pairs $\mathcal{O}\to F$ with $\mathrm{c}_1(F) = D$ and $\chi(F) = d+1-g = n$. As first noted by Kawai and Yoshioka, $\mathcal{C}_g^{[d]}$ is smooth, so a reasonable replacement for the Pandharipande-Thomas invariant is the topological

Euler characteristic $\chi(\mathcal{C}_g^{[d]})$. Let

$$Z_g^{PT}(y) = \sum_{d\geq 0}(-1)^{d+g}\chi(\mathcal{C}_g^{[d]})y^{n+g-1}$$

We then have a stable map-stable pair correspondence analogous to (1.3.1)

Theorem 1.5.1. *[MPT]* $Z_h^{GW}(u) = Z_h^{PT}(-e^{iu})$

1.6 Outline

The main aim of this thesis is to further explore the stable map-stable pair correspondence for K3 surfaces. We compute the Hodge polynomials of moduli spaces of higher rank stable pairs $\mathcal{O}^n \to F$ for all $n \geq 0$ and $r = \mathrm{rk}(F) \geq 0$, rederiving the $n = 1, r = 0$ calculation of [KY00]. We investigate the modularity of the resulting stable-pair potential functions.

The presentation is organized as follows. Chapters 2 and 3 review the theory of (semi)stable sheaves and stable pairs, respectively. In Chapter 4 we prove results about stable sheaves on K3 surfaces that will eventually dictate the geometry of the moduli spaces that allows for the computation of the Hodge polynomials. Chapter 5 treats some elementary aspects of "u-calculus" to allow for a more streamlined description of the computation in Chapter 6. We conclude in Chapter 7 with a discussion of further directions.

Chapter 2

Recollections on (Semi)stable Sheaves

Throughout the following k will be an algebraically closed characteristic 0 field, and by a sheaf on a scheme we will mean a coherent sheaf. Let X be a scheme over k of dimension n. In general the collection of all coherent sheaves on X cannot be represented by a scheme; a subcollection which can be reasonably algebraically parametrized by a scheme is picked out by introducing the notion of (semi)stability on the category $\mathrm{Coh}(X)$ of coherent sheaves on X. Stability structures can be defined on any abelian category [Rud97], and can even be extended to the derived category $D^b(X) = D^b\,\mathrm{Coh}(X)$ [Bri07], but for our purposes classical stability is sufficient. The question of whether the computation carries through for more general notions of stability will be addressed in Chapter 7. The following review of the basic properties of stable sheaves is adapted from [HL].

2.1 The Torsion Filtration

Let $\mathrm{Coh}(X)$ be the abelian category of coherent sheaves on X and $\mathrm{Coh}_d(X)$ the full subcategory of sheaves supported in dimension $\leq d$; by fiat $\mathrm{Coh}_{-1}(X)$ is the full subcategory with only the object $0 \in \mathrm{Coh}(X)$. The inclusion functor $i_d : \mathrm{Coh}_d(X) \to \mathrm{Coh}(X)$ has an obvious right adjoint t_d which to each $\mathcal{E} \in \mathrm{Ob}(\mathrm{Coh}(X))$ associates the subsheaf $T_d(\mathcal{E})$ of sections supported in dimension $\leq d$. Note that the adjunction map

$$i_d t_d(\mathcal{E}) \hookrightarrow \mathcal{E}$$

is injective. There is a torsion filtration for any $\mathcal{E} \in \mathrm{Coh}(X)$

$$0 = T_{-1}(E) \hookrightarrow T_0(\mathcal{E}) \hookrightarrow T_1(\mathcal{E}) \hookrightarrow \cdots \hookrightarrow T_{n-1}(\mathcal{E}) \hookrightarrow T_n(\mathcal{E}) = \mathcal{E} \tag{2.1}$$

It is the unique filtration by objects in $\mathrm{Coh}_d(X)$ for $d \leq n$. For $d' \leq d$ the i_d factor through fully faithful embeddings

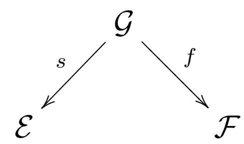
$$i_{d',d} = t_d i_{d'} : \mathrm{Coh}_{d'}(X) \to \mathrm{Coh}_d(X)$$

Definition 2.1.1. *The category* $\mathrm{Coh}_d^{\mathrm{pure}}(X)$ *of pure d-dimensional sheaves is the right-orthogonal complement of* $\mathrm{Coh}_{d-1}(X)$ *in* $\mathrm{Coh}_d(X)$*,* i.e. *the full subcategory of* $\mathrm{Coh}_d(X)$ *with objects*

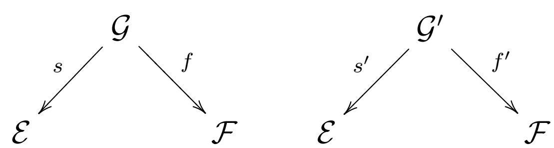
$$\{\mathcal{E} \in \mathrm{Ob}(\mathrm{Coh}_d(X)) | \operatorname{Hom}(\mathcal{F}, \mathcal{E}) = 0 \ \ \forall \mathcal{F} \in \mathrm{Ob}(\mathrm{Coh}_{d-1}(X))\}$$

In other words, a pure d-dimensional sheaf is a sheaf supported in dimension d none of whose sections has lower dimensional support. The ith quotient $T_i(\mathcal{E})/T_{i-1}(\mathcal{E})$ in (2.1) is pure of dimension i and (2.1) is the unique such filtration. The purity of a sheaf $\mathcal{E}$ is equivalent to the torsion filtration only having one nonzero quotient.

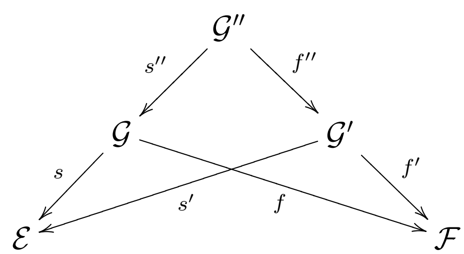
For $d' \leq d \leq n$, $\mathrm{Coh}_{d'}(X)$ forms a Serre subcategory of $\mathrm{Coh}_d(X)$; define $\mathrm{Coh}_{d',d}(X)$ to be the quotient abelian category of $\mathrm{Coh}_d(X)$ by $\mathrm{Coh}_{d'}(X)$, *cf.* [GM94]. Clearly $\mathrm{Coh}_{-1,d}(X) = \mathrm{Coh}_d(X)$. Recall that the objects of $\mathrm{Coh}_{d',d}(X)$ are objects of $\mathrm{Coh}_d(X)$ and a morphism $\mathcal{E} \to \mathcal{F}$ between objects $\mathcal{E}, \mathcal{F} \in \mathrm{Coh}_d(X)$ is an equivalence class of diagrams



where $\mathcal{G}$ is an object of $\mathrm{Coh}_d(X)$, $f \in \mathrm{Hom}_{\mathrm{Coh}_d(X)}(\mathcal{G}, \mathcal{F})$, and $s \in \mathrm{Hom}_{\mathrm{Coh}_d(X)}(\mathcal{G}, \mathcal{E})$ has kernel and cokernel in $\mathrm{Coh}_{d'}(X)$. Such a diagram is called a roof. Two roofs



are equivalent if there is a commutative diagram



where the top two morphisms are a roof.

For $d'' \leq d' \leq d$, the inclusions $i_d : \mathrm{Coh}_{d''}(X) \to \mathrm{Coh}_{d'}(X)$ yield fully faithful embeddings

$$i_{d'',d',d} : \mathrm{Coh}_{d'',d}(X) \to \mathrm{Coh}_{d',d}(X)$$

Definition 2.1.2. *The category of pure (d', d)-dimensional sheaves is the right orthogonal complement of* $\mathrm{Coh}_{d-1,d}(X)$ *in* $\mathrm{Coh}_{d',d}(X)$.

There is a unique filtration of any $\mathcal{E} \in \mathrm{Coh}_{d',d}(X)$

$$0 = T_{d'}(X) \hookrightarrow T_{d'+1}(X) \hookrightarrow \cdots \hookrightarrow T_{d-1}(\mathcal{E}) \hookrightarrow T_d(\mathcal{E}) = \mathcal{E} \tag{2.2}$$

by objects of $\mathrm{Coh}_{d',i}(X)$ for $d' \leq i \leq d$ whose ith quotient sheaf is purely (d', i)-

dimensional. The purity of $\mathcal{E} \in \mathrm{Ob}(\mathrm{Coh}_{d',d}(X))$ is equivalent to the condition that the torsion filtration (2.1) of $\mathcal{E}$ as an element of $\mathrm{Coh}(X)$ satisfies

$$T_{d'}(\mathcal{E}) = \cdots = T_{d-1}(\mathcal{E})$$

2.2 (Semi)stability

Let X be a projective scheme of dimension n with ample class H. For any sheaf $\mathcal{E}$ on X, the H-degree is

$$\deg \mathcal{E} = \mathrm{c}_1(\mathcal{E}).H^{n-1}$$

The H-Hilbert polynomial $P_{\mathcal{E}}$ is defined so that

$$P_{\mathcal{E}}(m) = \chi(\mathcal{E} \otimes H^m)$$

It is an elementary fact [Har77] that $P_{\mathcal{E}}$ is in fact a polynomial. $\deg \mathcal{E}, P_{\mathcal{E}}$ depend on the choice of ample class H, but we will suppress the dependence from the notation. Define $\alpha_i(\mathcal{E})$ by

$$P_{\mathcal{E}}(m) = \sum_{i \geq 0} \alpha_i(\mathcal{E}) \frac{m^i}{i!}$$

Clearly $\alpha_0(\mathcal{E}) = \chi(\mathcal{E})$ is the Euler characteristic. The first two coefficients in $P_{\mathcal{E}}$ are also easily determined:

Proposition 2.2.1. *Let $\mathcal{E}$ be a sheaf on X of dimension d*

1. *$P_{\mathcal{E}}$ has degree d, and $\alpha_d(\mathcal{E}) > 0$. If $d = n$, define*

$$\mathrm{rk}(\mathcal{E}) = \frac{\alpha_n(\mathcal{E})}{\alpha_n(\mathcal{O})}$$

2. *If $d = n$, then $\alpha_{n-1}(\mathcal{E}) = \deg \mathcal{E} - \frac{\deg K_X}{2}$ where K_X is the canonical class.*

For $\mathcal{E}$ locally free one can show $\mathrm{rk}(\mathcal{E})$ agrees with the normal notion of rank and $\deg \mathcal{E} = \deg \det \mathcal{E}$ with the normal notion of degree.

Remark 2.2.2. *If $\mathcal{F}$ is purely (d', d)-dimensional, any sheaf $\mathcal{E}$ which injects into $\mathcal{F}$ in $\mathrm{Coh}_{d',d}(X)$ must also be purely (d', d)-dimensional. In particular, $\alpha_d(\mathcal{E}) > 0$.*

The Hilbert polynomial is additive on short exact sequences, and therefore defines an additive homomorphism $P : K(\mathrm{Coh}(X)) \to \mathbb{Q}[m]_n$ from the Grothendieck group $K(\mathrm{Coh}(x))$ of $\mathrm{Coh}(X)$ into the additive group $\mathbb{Q}[m]_n$ of rational polynomials of degree $\leq n$. Similary, P restricts to a homomorphism $P : K(\mathrm{Coh}_d(X)) \to \mathbb{Q}[m]_d$, and if we let $\mathbb{Q}[m]_{d',d} = \mathbb{Q}[m]_d/\mathbb{Q}[m]_{d'}$, the Hilbert polynomial gives a homomorphism

$$P : K(\mathrm{Coh}_{d',d}(X)) \to \mathbb{Q}[m]_{d',d}$$

Note in particular that for $d = n$ and $d' = n - 2$,

$$\begin{array}{rccc} P: & K(\mathrm{Coh}_{n-2,n}(X)) & \longrightarrow & \mathbb{Q}[m]_{n-2,n} \\ & \mathcal{E} & \longmapsto & \alpha_n(\mathcal{E})\frac{m^n}{n!} + \left(\deg \mathcal{E} - \frac{\deg K_X}{2}\right)\frac{m^{n-1}}{(n-1)!} \end{array}$$

The normalized H-Hilbert polynomial of a sheaf $\mathcal{E}$ of dimension d is the monic polynomial

$$p_{\mathcal{E}} = \frac{d!}{\alpha_d(\mathcal{E})} P_{\mathcal{E}}$$

and similarly we get a map

$$p : \mathrm{Ob}(\mathrm{Coh}_{d',d}(X)) \to \mathbb{Q}[m]^{\mathrm{monic}}_{d',d}$$

where $\mathbb{Q}[m]^{\mathrm{monic}}_{d',d} \subset \mathbb{Q}[m]_{d',d}$ is the residues of monic polynomials. An ordering $<$ is defined between elements f, g of $\mathbb{Q}[m]^{\mathrm{monic}}_{d',d}$ as follows: $f < g$ if $f(n) < g(n)$ for $n >> 0$. Equivalently, $<$ is the lexicographic ordering on the coefficients. The

H-slope $\mu(\mathcal{E}) \in \mathbb{Q} \cup \{\infty\}$ is defined for any sheaf $\mathcal{E}$ as $\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\operatorname{rk}(\mathcal{E})}$—again, the H dependence is suppressed from the notation. In particular, as an ordered set, $\mathbb{Q}[m]^{\text{monic}}_{n-2,n}$ is isomorphic to $\mathbb{Q}$ with the standard ordering via

$$\mathbb{Q}[m]^{\text{monic}}_{n-2,n} \to \mathbb{Q} : m^n + nam^{n-1} \mapsto a + \frac{\deg K_X}{2}$$

So for $\mathcal{E}, \mathcal{F} \in \operatorname{Coh}_{n-2,n}(X)$, $p_{\mathcal{E}} \leq p_{\mathcal{F}}$ if and only if $\mu(\mathcal{E}) \leq \mu(\mathcal{F})$. We are now ready to define (semi)stability:

Definition 2.2.3. *A sheaf $\mathcal{E}$ on X is (d', d)-semistable (respectively (d', d)-stable) if it is purely (d', d)-dimensional and for any nontrivial injection $\mathcal{G} \hookrightarrow \mathcal{E}$ in $\operatorname{Coh}_{d',d}(X)$, $p_{\mathcal{G}} < p_{\mathcal{E}}$ ($p_{\mathcal{G}} \leq p_{\mathcal{E}}$). We will refer to $(-1, d)$-(semi)stability as Gieseker (semi)stability and $(n-2, n)$ (semi)stability as μ-(semi)stability.*

Remark 2.2.4. *An injection $\mathcal{E} \hookrightarrow \mathcal{F}$ in $\operatorname{Coh}_{d',d}(X)$ is* saturated *if the quotient is purely (d', d)-dimensional. For any injection $\mathcal{E} \hookrightarrow \mathcal{F}$ in $\operatorname{Coh}_{d',d}(X)$ the saturation is the minimal saturated subsheaf $\mathcal{E}' \hookrightarrow \mathcal{F}$ containing $\mathcal{E}$; it automatically satisfies $P_{\mathcal{E}} \leq P_{\mathcal{E}'}$ [HL]. It is easily seen that it is enough to test the condition for saturated subsheaves in the definition of (semi)stability .*

Remark 2.2.5. *For $d'' \leq d' \leq d \leq n$, we can make sense of the (d', d)-stability of a pure element $\mathcal{E} \in \operatorname{Ob}(\operatorname{Coh}_{d'',d}(X))$ by simply passing to $\operatorname{Coh}_{d',d}(X)$ via the projection $\operatorname{Coh}_{d'',d}(X) \to \operatorname{Coh}_{d'',d}(X)$. In that case, we clearly have*

$$(d', d) - \text{stable } \Rightarrow (d'', d) - \text{stable } \Rightarrow (d'', d) - \text{semistable } \Rightarrow (d', d) - \text{semistable}$$

For example, a pure sheaf $\mathcal{E} \in \operatorname{Ob}(\operatorname{Coh}_n(X))$ is μ-semistable (μ-stable) if for any injection $\mathcal{G} \hookrightarrow \mathcal{E}$ in $\operatorname{Coh}_{n-2,n}(X)$, $\mu(\mathcal{G}) \leq \mu(\mathcal{E})$ ($\mu(\mathcal{G}) < \mu(\mathcal{E})$). Since $\mathcal{E}$ is torsion free, every injection into $\mathcal{E}$ in $\operatorname{Coh}_{0,n}(X)$ is representable by an injection in $\operatorname{Coh}(X)$, and the projection of any injection in $\operatorname{Coh}(X)$ into $\operatorname{Coh}_{n-2,n}(X)$ is injective. Further, by

(2.2.4),

Proposition 2.2.6. *Let $\mathcal{E}$ be a pure n-dimensional sheaf on X. $\mathcal{E}$ is μ-semistable (μ-stable) in the above sense if and only if for any subsheaf $\mathcal{G} \hookrightarrow \mathcal{E}$ (in $\mathrm{Coh}(X)$) with $\mathrm{rk}(\mathcal{G}) < \mathrm{rk}(\mathcal{E})$, $\mu(\mathcal{G}) \leq \mu(\mathcal{E})$ ($\mu(\mathcal{G}) < \mu(\mathcal{E})$).*

2.3 Properties of (Semi)stable Sheaves

Here we quickly recount the usual progression of observations about (semi)stable sheaves. See [HL] for a more thorough treatment.

Remark 2.3.1. *Let $\mathcal{A}$ be an arbitrary abelian category. A morphism in $\mathcal{A}$ is said to be trivial if it is either 0 or an isomorphism.*

i. The Hilbert polynomial P is additive on short exact sequences in $\mathrm{Coh}_{d',d}(X)$, so for any such sequence of objects $\mathcal{E}, \mathcal{F}, \mathcal{G} \in \mathrm{Ob}(\mathrm{Coh}_{d',d}(X))$

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0 \tag{2.3}$$

we have

$$\alpha_d(\mathcal{E})\,(p_{\mathcal{E}} - p_{\mathcal{F}}) = \alpha_d(\mathcal{G})\,(p_{\mathcal{F}} - p_{\mathcal{G}}) \tag{2.4}$$

ii. From this and (2.2.2) it follows that $\mathcal{F} \in \mathrm{Ob}(\mathrm{Coh}_{d',d}(X))$ purely (d', d)-dimensional is (d', d)-(semi)stable if and only if for any nontrivial surjection $\mathcal{F} \twoheadrightarrow \mathcal{G}$ in $\mathrm{Coh}_{d',d}(X)$ with $\alpha_d(\mathcal{G}) > 0$, $p_{\mathcal{F}} < p_{\mathcal{G}}$ ($p_{\mathcal{F}} \leq p_{\mathcal{G}}$). Indeed, if $\mathcal{F}$ is (d', d)-(semi)stable, such a surjection gives a short exact sequence (2.3) and therefore the equation (2.4). By (2.2.2) we then have $p_{\mathcal{F}} < p_{\mathcal{G}}$ ($p_{\mathcal{F}} \leq p_{\mathcal{G}}$). Conversely, if $\mathcal{F}$ satisfies the above property, then for any nontrivial saturated injection $\mathcal{E} \hookrightarrow \mathcal{F}$, we again get (2.4) with $\alpha_d(\mathcal{G}) > 0$, and therefore $p_{\mathcal{E}} < p_{\mathcal{F}}$ $(p_{\mathcal{E}}) \leq p_{\mathcal{F}}$.

iii. Given stable $\mathcal{E}, \mathcal{F} \in \mathrm{Ob}(\mathrm{Coh}_{d',d}(X))$ with $p_{\mathcal{E}} = p_{\mathcal{F}}$, any morphism $\mathcal{E} \to \mathcal{F}$ in $\mathrm{Coh}_{d',d}(X)$ is trivial. Indeed, if $\mathcal{E}'$ is the image, we have a diagram

$$\mathcal{E} \twoheadrightarrow \mathcal{E}' \hookrightarrow \mathcal{F} \tag{2.5}$$

If one of the morphisms were nontrivial, either

$$p_{\mathcal{E}} < p_{\mathcal{E}'} \leq p_{\mathcal{F}} = p_{\mathcal{E}}$$

or

$$p_{\mathcal{E}} \leq p_{\mathcal{E}'} < p_{\mathcal{F}} = p_{\mathcal{E}}$$

which is nonsense.

Definition 2.3.2. *On any partially ordered set $(S, \leq)$, $y \in S$ is an immediate successor to $x \in S$ if $x < y$ and there is no $y' \in S$ such that*

$$x < y' < y$$

iv. Given stable $\mathcal{E}, \mathcal{F} \in \mathrm{Ob}(\mathrm{Coh}_{d',d}(X))$ with $p_{\mathcal{F}}$ an immediate successor to $p_{\mathcal{E}}$ in $\mathbb{Q}[m]_{d',d}$, any morphism $\mathcal{E} \to \mathcal{F}$ in $\mathrm{Coh}_{d',d}(X)$ is either injective or surjective. Such a morphism factors as in (2.5)

$$\mathcal{E} \twoheadrightarrow \mathcal{E}' \hookrightarrow \mathcal{F}$$

one of which must be trivial since otherwise

$$p_{\mathcal{E}} < p_{\mathcal{E}'} < p_{\mathcal{F}}$$

v. For any semistable $\mathcal{E}, \mathcal{F} \in \mathrm{Ob}(\mathrm{Coh}_{d',d}(X))$ with $p_{\mathcal{E}} > p_{\mathcal{F}}$, $\mathrm{Hom}_{\mathrm{Coh}_{d',d}(X)}(\mathcal{E}, \mathcal{F}) =$

0. Indeed, a nonzero morphism would factor nontrivially as in (2.5)

$$\mathcal{E} \twoheadrightarrow \mathcal{E}' \hookrightarrow \mathcal{F}$$

in which case

$$p_{\mathcal{E}} \leq p_{\mathcal{E}'} \leq p_{\mathcal{F}}$$

vi. For any $\mathcal{E} \in \mathrm{Ob}(\mathrm{Coh}_{d',d}(X))$ purely (d', d)-dimensional, there is a unique filtration

$$0 = HN_{-1}(\mathcal{E}) \hookrightarrow HN_0(\mathcal{E}) \hookrightarrow \cdots \hookrightarrow HN_{\ell-1}(\mathcal{E}) \hookrightarrow HN_\ell(\mathcal{E}) = \mathcal{E} \qquad (2.6)$$

whose quotients

$$Q_i^{HN}(\mathcal{E}) = HN_i(\mathcal{E})/HN_{i-1}(\mathcal{E})$$

are (d', d)-semistable with

$$p_{Q_0^{HN}(\mathcal{E})} > \cdots > p_{Q_\ell^{HN}(\mathcal{E})}$$

It is called the Harder-Narasimhan filtration.

vii. For any $\mathcal{E} \in \mathrm{Ob}(\mathrm{Coh}_{d',d}(X))$ purely (d', d)-dimensional, there is a unique injection

$$\mathcal{G} \hookrightarrow \mathcal{E}$$

in $\mathrm{Coh}_{d',d}(X)$ such that $p_{\mathcal{G}} \geq p_{\mathcal{E}}$, and $\mathcal{G}$ is maximal with respect to this property, *i.e.* for any nontrivial injection $\mathcal{F} \hookrightarrow \mathcal{E}$, $p_{\mathcal{G}} \geq p_{\mathcal{F}}$, and if $p_{\mathcal{F}} = p_{\mathcal{G}}$ then the injection factors $\mathcal{F} \hookrightarrow \mathcal{G}$. $\mathcal{G}$ is called the *maximal destabilizing subsheaf*; it is (d', d)- semistable, and in fact equal to the first piece of the Harder-Narasimhan filtration,

$$\mathcal{G} = HN_0(\mathcal{E}) \hookrightarrow \mathcal{E}$$

viii. Dually, for any $\mathcal{E} \in \mathrm{Ob}(\mathrm{Coh}_{d',d}(X))$ purely (d',d)-dimensional, there is a unique surjection

$$\mathcal{E} \twoheadrightarrow \mathcal{G}$$

in $\mathrm{Coh}_{d',d}(X)$ such that $p_{\mathcal{E}} \geq p_{\mathcal{G}}$ and $\mathcal{G}$ is maximal with respect to this property, *i.e.* for any nontrivial surjection $\mathcal{E} \twoheadrightarrow \mathcal{F}$, $p_{\mathcal{F}} \geq p_{\mathcal{G}}$ and if $p_{\mathcal{F}} = p_{\mathcal{G}}$ then the surjection factors $\mathcal{G} \twoheadrightarrow \mathcal{F}$. In fact it is the last quotient of the Harder-Narasimhan filtration (2.6).

$$\mathcal{E} \twoheadrightarrow Q_\ell^{HN}(\mathcal{E})$$

ix. For any (d',d)-semistable sheaf $\mathcal{E} \in \mathrm{Ob}(\mathrm{Coh}_{d',d}(X))$, there is a Jordan-Hölder filtration

$$0 = JH_{-1}(\mathcal{E}) \subset JH_0(\mathcal{E}) \subset \cdots \subset JH_{k-1}(\mathcal{E}) \subset JH_k(\mathcal{E}) = \mathcal{E} \tag{2.7}$$

in $\mathrm{Coh}_{d',d}(X)$ whose quotients $Q_i^{JH}(\mathcal{E}) = JH_i(\mathcal{E})/JH_{i-1}(\mathcal{E})$ are (d',d)-stable and all of whose normalized Hilbert polynomials are equal,

$$p_{\mathcal{E}} = p_{Q_0^{JH}(\mathcal{E})} = \cdots = p_{Q_k^{JH}(\mathcal{E})}$$

(2.7) is not unique, but any such filtration will have isomorphic associated graded module

$$\mathrm{gr}^{JH}(\mathcal{E}) = \bigoplus_i Q_i^{JH}(\mathcal{E})$$

Definition 2.3.3. *Two (d',d)-semistable sheaves $\mathcal{E}, \mathcal{F}$ are S-equivalent if*

$$\mathrm{gr}^{JH}(\mathcal{E}) \cong \mathrm{gr}^{JH}(\mathcal{F})$$

2.4 Moduli of (Semi)stable Sheaves

We restrict our attention for the moment to Gieseker stability; throughout this section, (semi)stability will mean Gieseker (semi)stability.

Let X be a smooth projective k-scheme, S an arbitrary k-scheme.

Definition 2.4.1. *A family of (semi)stable sheaves on $X \times S/S$ is a sheaf $\mathcal{E}$ on $X \times S$ flat over S such that for each k-point p of S the pullback $\mathcal{E}_p$ to the fiber X_p is a (semi)stable sheaf on X_p. Two families $\mathcal{E}, \mathcal{F}$ on $\pi : X \times S \to S$ are equivalent, $\mathcal{E} \sim \mathcal{F}$, if for some line bundle $\mathcal{L}$ on S there is an isomorphism $\mathcal{E} \cong \mathcal{F} \otimes \pi^*\mathcal{L}$. The moduli of semistable sheaves functor $\mathcal{M} : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Sets}$ is*

$$\mathcal{M}(S) = \{\text{families } \mathcal{E} \text{ of semistable sheaves on } X \times S/S\}/\sim$$

and the value of $\mathcal{M}$ on a morphism $T \to S$ is pullback along the resulting map $X \times T \to X \times S$.

Recall that a fine moduli space for a functor $F : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Sets}$ is a scheme M whose functor of points $\mathrm{Hom}(\,\cdot\,, M) : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Sets}$ is isomorphic to F; in that case F is said to be representable. In particularly, representability requires the existence of a family $\mathcal{F}$ on $X \times M$—corresponding to the identity morphism $M \to M$—such that any family $\mathcal{F}'$ on $X \times S/S$ is equivalent to the pullback of $\mathcal{F}$ along the classifying morphism $S \to M$ arising from the identification $\mathrm{Hom}(\,\cdot\,, M) \cong F$. $\mathcal{F}$ is called a universal family.

It is well-known that $\mathcal{M}$ is not in general representable. Whenever there is a properly semistable sheaf $\mathcal{E}$, $\mathcal{E}$ has a Jordan-Hölder filtration by stable sheaves $\mathcal{E}_i$ with the same Hilbert polynomial, and a family of sheaves on X can be constructed over $\mathbb{A}^1$ which is $\mathcal{E}$ generically, but which collapses the filtration over 0 yielding $\bigoplus_i \mathcal{E}_i$

as a fiber. For example, take a nontrivial extension

$$0 \to \mathcal{E}_0 \to \mathcal{E} \to \mathcal{E}_1 \to 0$$

with $\mathcal{E}_0, \mathcal{E}_1$ stable and $p_{\mathcal{E}} = p_{\mathcal{E}_0} = p_{\mathcal{E}_1}$; let $e \in \mathrm{Ext}^1(\mathcal{E}_0, \mathcal{E}_1)$ be the extension class. There is a universal extension

$$0 \to \pi^* \mathcal{E}_0 \to \mathcal{G} \to \pi^* \mathcal{E}_1 \to 0 \tag{2.8}$$

on $X \times \mathbb{E}\mathrm{xt}^1(\mathcal{E}_1, \mathcal{E}_0)$, where $\mathbb{E}\mathrm{xt}^1(\mathcal{E}_1, \mathcal{E}_0) = \mathrm{Spec}\ k[\mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_0)]$ and $\pi : X \times \mathbb{E}\mathrm{xt}^1(\mathcal{E}_1, \mathcal{E}_0) \to X$ is the projection. $\mathcal{G}$ is flat over $\mathbb{E}\mathrm{xt}^1(\mathcal{E}_1, \mathcal{E}_0)$, and the restriction of (2.8) to any fiber $X \times f$ is the extension corresponding to f. If $\mathcal{M}$ can be represented by M, then the line through 0 and e would yield a morphism $\mathbb{A}^1 \to M$ sending $\mathbb{A}^1 \setminus 0$ to the point $[\mathcal{F}] \in M$, but 0 to $[\bigoplus_i \mathcal{F}_i]$, which is clearly impossible.

Thus, any scheme M representing families of semistable sheaves must identify properly semistable sheaves with the same Jordan-Hölder constituents—that is, S-equivalent semistable sheaves. This is the only obstruction to representing $\mathcal{M}$ in the following weak sense:

Definition 2.4.2. *A k-scheme M is a* coarse moduli space *for a functor $F : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Sets}$ if there is a natural transformation $F \to \mathrm{Hom}(\,\cdot\,, M)$ and M is universal with respect to this property. Precisely, for any k-scheme M' and natural transformation $F \to \mathrm{Hom}(\,\cdot\,, M')$ there is a unique morphism $M \to M'$ such that*

$$\begin{array}{ccc} F & \longrightarrow & \mathrm{Hom}(\,\cdot\,, M) \\ & \searrow & \downarrow \\ & & \mathrm{Hom}(\,\cdot\,, M') \end{array}$$

commutes.

Theorem 2.4.3. *There is a coarse moduli space M for the functor $\mathcal{M}$. k-points of M are S-equivalence classes of semistable sheaves on X.*

Recall that for any sheaf $\mathcal{E}$ on X, the Mukai vector $v(\mathcal{E}) \in H^*(X, \mathbb{Z})$ is

$$v(\mathcal{E}) = \mathrm{ch}(\mathcal{E})\sqrt{\mathrm{Td}(X)}$$

(see Section 4.1). Fix a Mukai vector $v \in H^*(X, \mathbb{Z})$. Letting $M(v) \subset M$ be the open set with closed points corresponding to sheaves $\mathcal{E}$ with $v(\mathcal{E}) = v$, we further have

Theorem 2.4.4. *$M(v)$ is projective.*

If we restrict our attention to stable sheaves, $\mathcal{M}$ still has a chance at being representable. The open subfunctor $\mathcal{M}^s$ of $\mathcal{M}$ is defined by

$$\mathcal{M}^s(S) = \{\text{families } \mathcal{E} \text{ of stable sheaves on } X \times S/S\}/\sim$$

Also let $M^s \subset M$ be the open set with closed points corresponding to stable sheaves. M^s will in general not have a universal family, but there is always a quasi-universal family:

Theorem 2.4.5. *There is a sheaf $\mathcal{F}$ on $X \times M^s$ flat over M^s such that for any family of stable sheaves $\mathcal{E}$ on $X \times S$, $\mathcal{E} \otimes \pi^*\mathcal{V} \cong f^*\mathcal{F}$, where $\pi : X \times S \to S$ is the projection, $\mathcal{V}$ is a locally free sheaf on S, and $f : X \times S \to X \times M^s$ is the base-change of the morphism guaranteed by the universal property of M^s.*

Remark 2.4.6. *The important fact we will need is that étale locally on M^s there always exists a universal family.*

Chapter 3

Recollections on Coherent Systems

Here we review the theory of coherent systems as developed by [He98]; see also [LP93, LP95]. Naively, a coherent system is a sheaf together with a choice of n sections. We first define coherent systems, briefly outline the basic properties and deformation theory, and conclude by discussing the moduli of coherent systems.

3.1 Definitions

Let X, S be schemes over k.

Definition 3.1.1. *A* coherent system[1] $(\mathcal{E}, U)$ *of level n on X is a sheaf $\mathcal{E}$ on X, a vector space U with* $\dim U = n$*, and a morphism $U \otimes \mathcal{O} \to \mathcal{E}$. A morphism of coherent systems $(\mathcal{E}, U) \to (\mathcal{F}, V)$ is a commutative diagram*

$$\begin{array}{ccc} U \otimes \mathcal{O} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ V \otimes \mathcal{O} & \longrightarrow & \mathcal{F} \end{array}$$

To develop the moduli of coherent systems, we need an appropriate notion of

[1] We will often refer to coherent systems simply as the pair of sheaves and suppress the structure maps from the notation.

families of coherent systems. The first step is to understand the relative situation:

Definition 3.1.2. *A* relative coherent system $\Lambda = (\mathcal{E}, \mathcal{U})$ *on* $f : X \to S$ *is a sheaf* $\mathcal{E}$ *on* X*, a sheaf* $\mathcal{U}$ *on* S*, and a morphism* $f^*\mathcal{U} \to \mathcal{E}$*. A morphism of relative coherent systems* $(\mathcal{E}, \mathcal{U}) \to (\mathcal{F}, \mathcal{V})$ *is a commutative diagram*

$$\begin{array}{ccc} f^*\mathcal{U} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ f^*\mathcal{V} & \longrightarrow & \mathcal{F} \end{array}$$

We will often refer to relative coherent systems Λ on X/S simply as coherent systems on X/S. Of course, a relative coherent system on X/k is just a coherent system on X.

Relative coherent systems form an abelian category $\mathrm{CohSys}(X/S)$.

Lemma 3.1.3. CohSys(X/S) *has enough injectives.*

Proof. See [He98]. □

Given a coherent system $\Lambda = (\mathcal{E}, \mathcal{U})$ on X/S and a diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{h} & S \end{array}$$

define $g^*\Lambda = (g^*\mathcal{E}, h^*\mathcal{U})$ with the obvious structure map, $f'^*(h^*\mathcal{U}) \cong g^*(f^*\mathcal{U}) \to g^*\mathcal{E}$. In particular, for any open $U \subset S$,

$$\begin{array}{cccc} X \times_S U & \longrightarrow X_U & \xrightarrow{g} & X \\ & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ & U & \xrightarrow{h} & S \end{array}$$

Define the restriction $\Lambda|_U = g^*\Lambda$. For any relative coherent systems Λ, Λ' on X/S, the association $U \mapsto \mathrm{Hom}(\Lambda|_U, \Lambda'|_U)$ for open affine U defines a coherent $\mathcal{O}_S$-module $\mathcal{H}om(\Lambda, \Lambda')$ whose global sections over S are $\mathrm{Hom}(\Lambda, \Lambda')$.

Lemma 3.1.4. *[He98] For any coherent system Λ on X/S,*

$$\mathcal{H}om(\Lambda, \cdot\,) : \mathrm{CohSys}(X/S) \to \mathrm{Coh}(S)$$

is left exact. It's derived functor is denoted by $R\mathcal{H}om(\Lambda, \cdot\,) : D\mathrm{CohSys}(X/S) \to D(S)$; the cohomology sheaves are $R\mathcal{H}om(\Lambda, \cdot\,)$ of denoted by $\mathcal{E}xt^i(\Lambda, \cdot\,)$.

The $\mathcal{E}xt^i(\Lambda, \cdot\,)$ are related to the standard $\mathcal{E}xt$ groups by the following lemma

Lemma 3.1.5. *[He98] For any coherent systems $\Lambda = (\mathcal{E}, \mathcal{U}), \Lambda' = (\mathcal{E}', \mathcal{U}')$ on $f : X \to S$ the obvious sequence of $\mathcal{O}_S$ modules*

$$0 \to \mathcal{H}om(\Lambda, \Lambda') \to \mathcal{H}om(\mathcal{U}, \mathcal{U}') \oplus f_*\mathcal{H}om(\mathcal{E}, \mathcal{E}') \to \mathcal{H}om(\mathcal{U}, f_*\mathcal{E}')$$

is exact. Furthermore, if Λ' is injective, the rightmost map is surjective.

Corollary 3.1.6. *There is a triangle in the derived category $D\mathrm{CohSys}(X)$*

$$R\mathcal{H}om(\Lambda, \Lambda') \to R\mathcal{H}om(\mathcal{U}, \mathcal{U}') \oplus Rf_*R\mathcal{H}om(\mathcal{E}, \mathcal{E}') \to R\mathcal{H}om(\mathcal{U}, Rf_*\mathcal{E}') \to R\mathcal{H}om(\Lambda, \Lambda')[1] \tag{3.1}$$

3.2 Deformation Theory of Coherent Systems

Let X be a smooth k-scheme and fix a coherent system $\Lambda = (\mathcal{E}, U)$ on X. By [He98], the deformation space of Λ is naturally $\mathrm{Ext}^1(\Lambda, \Lambda)$, and the obstruction lies in the kernel of the composition

$$\mathrm{Ext}^1(\Lambda, \Lambda) \to \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \xrightarrow{\mathrm{tr}} \mathrm{Hom}(\mathcal{O}, \mathcal{O})$$

Over a point, (3.1) simply becomes a triangle

$$R\operatorname{Hom}(\Lambda,\Lambda) \to \operatorname{End}(U) \oplus R\operatorname{Hom}(\mathcal{E},\mathcal{E}) \to R\operatorname{Hom}(U\otimes\mathcal{O},\mathcal{E}) \to R\operatorname{Hom}(\Lambda,\Lambda)[1]$$

in $D^b(k)$, the bounded derived category of k-vector spaces. Here $\operatorname{End}(U)$ is supported in degree 0, so $R\operatorname{Hom}(\Lambda,\Lambda)[1]$ is a cone of the morphism $R\operatorname{Hom}(\mathcal{E},\mathcal{E}) \to R\operatorname{Hom}(U\otimes\mathcal{O},\mathcal{E})$ above degree 0. Formally, by the octahedral axiom there is a diagram

$$\begin{array}{ccccccc}
 & & \operatorname{End}(U) & = & \operatorname{End}(U) & & \\
 & & \uparrow & & \uparrow & & \\
R\operatorname{Hom}(U\otimes\mathcal{O},\mathcal{E})[-1] & \longrightarrow & R\operatorname{Hom}(\Lambda,\Lambda) & \longrightarrow & \operatorname{End}(U)\oplus R\operatorname{Hom}(\mathcal{E},\mathcal{E}) & \longrightarrow & R\operatorname{Hom}(U\otimes\mathcal{O},\mathcal{E}) \\
\| & & \uparrow & & \uparrow & & \| \\
R\operatorname{Hom}(U\otimes\mathcal{O},\mathcal{E})[-1] & \longrightarrow & C & \longrightarrow & R\operatorname{Hom}(\mathcal{E},\mathcal{E}) & \longrightarrow & R\operatorname{Hom}(U\otimes\mathcal{O},\mathcal{E}) \\
 & & \uparrow & & \uparrow & & \\
 & & \operatorname{End}(U)[-1] & = & \operatorname{End}(U)[-1] & &
\end{array} \tag{3.2}$$

Whose columns and rows are triangles. If we let $x \in D^b(X)$ be the two-term complex

$$x = [U \otimes \mathcal{O} \to \mathcal{E}]$$

with $\mathcal{E}$ placed in degree 1, there is also a triangle

$$x \to U\otimes\mathcal{O} \to \mathcal{E} \to x[1]$$

and applying $R\operatorname{Hom}(\cdot,\mathcal{E})$ there is another

$$R\operatorname{Hom}(x,\mathcal{E})[-1] \to R\operatorname{Hom}(\mathcal{E},\mathcal{E}) \to R\operatorname{Hom}(U\otimes\mathcal{O},\mathcal{E}) \to R\operatorname{Hom}(x,\mathcal{E})$$

Thus, $C[1]$ in (3.2) and $R\operatorname{Hom}(x,\mathcal{E})$ are both cones of the same morphism $R\operatorname{Hom}(\mathcal{E},\mathcal{E}) \to R\operatorname{Hom}(U\otimes\mathcal{O},\mathcal{E})$ and are therefore isomorphic. By the long exact cohomology sequence associated to the first vertical triangle in (3.2), the deformation space of Λ is $\mathrm{Ext}^1(\Lambda,\Lambda) \cong \mathrm{Hom}(x,\mathcal{E})$ and the obstruction lies in the kernel of the composition of the canonical map

$$\mathrm{Ext}^1(x,\mathcal{E}) \to \mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \xrightarrow{\mathrm{tr}} \mathrm{Ext}^2(\mathcal{O},\mathcal{O})$$

3.3 Stable Pairs and Their Moduli

Throughout this section, by (semi)stability we will again mean Gieseker (semi)stability.

Definition 3.3.1. *A level n stable pair on X is a level n coherent system $(\mathcal{E},U)$ on X such that $\mathcal{E}$ is stable and the map $U \otimes \mathcal{O} \to \mathcal{E}$ is injective on global sections. Similarly, a family of stable pairs $(\mathcal{E},\mathcal{U})$ on $f : X \to S$ is a relative coherent system such that $\mathcal{E}$ is a family of stable sheaves, $\mathcal{U}$ is locally free, and the map $\mathcal{U} \to f_*\mathcal{E}$ is injective. Equivalently, a family of stable pairs is a relative coherent system $(\mathcal{E},\mathcal{U})$ such that is $\mathcal{E}$ flat over S, $\mathcal{U}$ is locally free, and the restriction to every fiber is a stable pair.*

The main result of [He98] is the existence of a projective moduli space Syst^n of level n stable pairs $(\mathcal{E},U)$. More precisely, define a moduli functor $F^n : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Sets}$ by

$$F^n(S) = \{\text{families of level n stable pairs}\}/\cong$$

where $\cong$ is isomorphism as relative coherent systems. Then

Theorem 3.3.2. *[He98] There is a coarse moduli space Syst^n for F. If M is the moduli of stable sheaves, there is a canonical forgetful morphism $p : \mathrm{Syst}^n \to M$ sending a stable pair $(\mathcal{E},U)$ to the sheaf $\mathcal{E}$. For $v \in H^*(X,\mathbb{Z})$, the open subscheme $\mathrm{Syst}^n(v) \subset \mathrm{Syst}^n$ with closed points corresponding to stable pairs $(\mathcal{E},U)$ with $v(\mathcal{E}) = v$ is projective.*

Chapter 4

Stable Sheaves on K3 Surfaces

4.1 Characteristic Classes

Let X be a smooth surface over k. Recall that the Chern roots of a sheaf $\mathcal{E}$ on X are cohomology classes $x_i \in H^2(X, \mathbb{Z})$ such that the total Chern class $\mathrm{c}(\mathcal{E})$ factors as

$$\mathrm{c}(\mathcal{E}) = \prod_i (1 + x_i)$$

The Todd class $\mathrm{Td}(\mathcal{E}) \in H^*(X, \mathbb{Z})$ of $\mathcal{E}$ is then

$$\mathrm{Td}(\mathcal{E}) = \prod_i \frac{x_i}{1 - e^{-x_i}} \tag{4.1}$$

The Todd class $\mathrm{Td}(\mathcal{T}_X)$ of the tangent sheaf of X is referred to simply as the Todd class $\mathrm{Td}(X)$ of X. Explicitly

$$\mathrm{Td}(\mathcal{E}) = 1 + \frac{\mathrm{c}_1(\mathcal{E})}{2} + \frac{\mathrm{c}_1(\mathcal{E})^2 + \mathrm{c}_2(\mathcal{E})}{12}$$

and thus

$$\mathrm{Td}(X) = 1 - \frac{K}{2} + \left(\frac{K^2 + e(X)}{12}\right)\omega$$

where $K = c_1(\Omega^1_X)$ is the canonical class of X, $\omega \in H^4(X, \mathbb{Z})$ is the Poincaré dual of the point class generator of $H_0(X, \mathbb{Z})$, and $\chi(X)$ is the topological Euler characteristic of X. Indeed, $\chi(X) = \int_{[X]} c_2(\mathcal{T}_X)$, so $c_2(\mathcal{T}_X) = \chi(X)\omega$.

For any sheaf $\mathcal{E}$ on X, the Mukai vector $v(\mathcal{E}) \in H^*(X, \mathbb{Z})$ of $\mathcal{E}$ is defined by

$$v(\mathcal{E}) = \operatorname{ch} \mathcal{E} \sqrt{\operatorname{Td}(X)}$$

Given an element $v = v_0 + v_1 + v_2 \in H^*(X, \mathbb{Z})$, where $v_i \in H^{2i}(X, \mathbb{Z})$ define $v^\vee = v_0 - v_1 + v_2$, so $v(\mathcal{E}^\vee) = v(\mathcal{E})^\vee$, where $\mathcal{E}^\vee \in D^b(X)$ is the derived dual of $\mathcal{E}$. We will often write an element $v \in H^*(X, \mathbb{Z})$ as $v = (r, D, a)$, where $r, a \in \mathbb{Z}$ and $D \in H^2(X, \mathbb{Z})$ by identifying $\mathbb{Z} \cong H^0(X, \mathbb{Z})$ using the fundamental class $[X] \in H_4(X, \mathbb{Z})$, and $\mathbb{Z} \cong H^4(X, \mathbb{Z})$ using the generator $\omega \in H^4(X, \mathbb{Z})$ Poincaré dual to the canonical generator of $H_0(X, \mathbb{Z})$.

The Mukai pairing on two elements $v, w \in H^*(X, \mathbb{Z})$ is defined by

$$(v, w) = -\int_{[X]} v^\vee w = \int_{[X]} (v_1 w_1 - v_0 w_2 - v_2 w_0)$$

In particular if $v = (r, D, a)$,

$$(v, v) = D^2 - 2ra = 2g - 2 - 2ra$$

where g is the arithmetic genus of a curve in the divisor class D, $D^2 = 2g - 2$. By Grothendieck-Riemann-Roch,

$$\chi(\mathcal{E}) = \int_{[X]} \operatorname{ch}(\mathcal{E}) \operatorname{Td}(X)$$

and since $R\operatorname{Hom}(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^\vee \otimes^L \mathcal{F}$ in $D^b(X)$ for any two sheaves $\mathcal{E}, \mathcal{F}$ on X,

$$(v(\mathcal{E}), v(\mathcal{F})) = -\chi(R\operatorname{Hom}(\mathcal{E}, \mathcal{F}))$$

If X is K-trivial,

$$\mathrm{Td}(X) = 1 + \frac{e(X)}{12}\omega$$

so in particular for X a K3

$$\sqrt{\mathrm{Td}(X)} = 1 + \omega$$

Further, given a sheaf $\mathcal{E}$,

$$\mathrm{ch}(\mathcal{E}) = \mathrm{rk}(\mathcal{E}) + \mathrm{c}_1(\mathcal{E}) + \frac{\mathrm{c}_1(\mathcal{E})^2 - 2\mathrm{c}_2(\mathcal{E})}{2}$$

so

$$\chi(\mathcal{E}) = \int_{[X]} \left(2\,\mathrm{rk}(\mathcal{E})\omega + \frac{\mathrm{c}_1(\mathcal{E})^2}{2} - \mathrm{c}_2(\mathcal{E})\right)$$

$$\begin{aligned} v(\mathcal{E}) &= \mathrm{rk}(\mathcal{E}) + \mathrm{c}_1(\mathcal{E}) + \left(\mathrm{rk}(\mathcal{E})\omega + \frac{\mathrm{c}_1(\mathcal{E})^2}{2} - \mathrm{c}_2(\mathcal{E})\right) \\ &= (\mathrm{rk}(\mathcal{E}), \mathrm{c}_1(\mathcal{E}), \chi(\mathcal{E}) - \mathrm{rk}(\mathcal{E})) \end{aligned}$$

in the above notation.

4.2 Moduli of Sheaves on K3 Surfaces

The moduli spaces of Gieseker stable sheaves on K-trivial surfaces are particularly well behaved. The first important observation is that the stable loci are smooth.

Proposition 4.2.1. *Let X be a K-trivial surface, M the moduli space of Gieseker semistable sheaves on X, and $M^s \subset M$ the stable locus. Then M^s is smooth.*

Proof. By the deformation theory of sheaves [HL], the Zariski tangent space to M at a point $[\mathcal{E}] \in M^s$ is $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$ and the local obstruction lies in $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E})_0$, *i.e.* the kernel of the trace map $\mathrm{tr} : \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) \to \mathrm{Ext}^2(\mathcal{O}, \mathcal{O})$. By Serre duality, tr is dual to the identity section

$$\mathrm{id} : \mathrm{Hom}(\mathcal{O}, \mathcal{O}) \to \mathrm{Hom}(\mathcal{E}, \mathcal{E}) \tag{4.2}$$

but since $\mathcal{E}$ is stable it is simple by (2.3.*iii*), and therefore (4.2) is an isomorphism. Thus, M is smooth at $[\mathcal{E}]$, and since M^s is open in M, M^s is smooth. □

For a given Mukai vector $v = (r, D, a) \in H^*(X, \mathbb{Z})$, let $M(v) \subset M$ be the open set of points corresponding to sheaves $\mathcal{E}$ with $v(\mathcal{E}) = v$, and $M^s(v) = M(v) \cap M^s$. Suppose D has genus g, *i.e.* $D^2 = 2g - 2$. $M^s(v)$ is smooth, and the tangent space to a point $[\mathcal{E}] \in M^s(v)$ is canonically $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$, so the dimension of $M^s(v)$ is

$$\dim M^s(r, D, a) = 2 - \chi(R\,\mathrm{Hom}(\mathcal{E}, \mathcal{E})) = 2 + (v, v) = 2g - 2ra \tag{4.3}$$

By the main existence theorem (2.4.3), the moduli spaces $M^s(v)$ are projective if $M(v) = M^s(v)$—that is, if there are no properly semistable sheaves. This will be the case when D, the component of v lying in $H^2(X, \mathbb{Z})$, is of minimal degree:

Definition 4.2.2. *A divisor class $D \in \mathrm{Pic}(X)$ has minimal degree if no positive line bundle has smaller intersection product with H, that is*

$$D.H = \min\{L.H | L \in \mathrm{Pic}(X), L.H > 0\}$$

Remark 4.2.3. *Note that for any divisor class D of minimal degree, every divisor in $|D|$ is reduced and irreducible.*

Examples 4.2.4.

1. *If X has Picard rank one and H is the ample generator, then $D = H$ has minimal degree.*

2. *If X is an elliptic K3 surface with section, $\mathrm{Pic}(X) = \mathbb{Z}\sigma \oplus \mathbb{Z}f$, where f is the fiber class and σ the section class. Choosing $H = \sigma + 3f$ to be the ample class,*

we have

$$(a\sigma + bf).H = a + b$$

So σ and f are clearly of minimal degree, since both have intersection product 1 with H.

Assume the divisor class D on X is of minimal degree, and let $v = (r, D, a) \in H^*(X, \mathbb{Z})$. In this case there are no properly μ-semistable sheaves:

Lemma 4.2.5. *Every 2-dimensional μ-semistable sheaf $\mathcal{E}$ with $v(\mathcal{E}) = v$ as above is μ-stable.*

Proof. Otherwise there is a nontrivial injection $\varphi : \mathcal{F} \hookrightarrow \mathcal{E}$ in $\mathrm{Coh}(X)$ with $\mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$ and $\mu(\mathcal{F}) = \mu(\mathcal{E})$, *cf.* (2.2.6). We must then have $\deg \mathcal{F} > \deg \mathcal{E}$, contradicting the minimal degree of D. □

In particular, by (2.2.4) this implies that μ-stability is equivalent to Gieseker stability for 2-dimensional sheaves $\mathcal{E}$ with $\mathrm{c}_1(\mathcal{E})$ of minimal degree, and every Gieseker semistable sheaf with $\mathrm{c}_1(\mathcal{E}) = D$ is Gieseker stable. We have a similar result for 1-dimensional sheaves:

Lemma 4.2.6. *Every pure 1-dimesional sheaf $\mathcal{E}$ with $v(\mathcal{E}) = v$ as above is stable.*

Proof. Note that the Hilbert polynomial of a 1-dimensional sheaf $\mathcal{E}$ is $P_{\mathcal{E}} = (\deg \mathcal{E})m + \chi(\mathcal{E})$. By (4.2.3) the support $\operatorname{Supp} \mathcal{E}$ of $\mathcal{E}$ is reduced and irreducible, and $\mathcal{E}$ is rank one on $\operatorname{Supp} \mathcal{E}$. For any nontrivial subsheaf $\mathcal{G} \hookrightarrow \mathcal{E}$, the quotient $\mathcal{Q}$ is 0-dimensional and $\deg \mathcal{E} = \deg \mathcal{G}$, so

$$p_{\mathcal{G}} = m + \frac{\chi(\mathcal{G})}{\deg \mathcal{G}} < p_{\mathcal{E}} = m + \frac{\chi(\mathcal{E})}{\deg \mathcal{E}}$$

since $\chi(\mathcal{E}) - \chi(\mathcal{G}) = \chi(\mathcal{Q}) > 0$. □

Lemma 4.2.7. *Suppose for some Mukai vector v, $M(v) = M^s(v)$. Then $M(v)$ is irreducible.*

Proof. See [HL]. □

Finally, by deforming X one can show that $M(v)$ is in fact nonempty whenever the dimension count from (4.3) is nonnegative. Thus,

Proposition 4.2.8. *Let X be a K3 surface, D a divisor class of minimal degree. For $r \geq 0, g \geq ra$, $M(r, D, a)$ is a smooth projective irreducible scheme of dimension $2g - 2ra$.*

The tangent space to a point $[\mathcal{E}] \in M(v)$ is canonically $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$, and Serre duality gives a nondegenerate pairing

$$\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \times \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \to \mathrm{Ext}^2(\mathcal{O}, \mathcal{O}) \cong k$$

The pairing can be shown to give $M(v)$ the structure of an irreducible symplectic manifold:

Definition 4.2.9. *An irreducible symplectic variety X is a projective simply connected variety such that $H^0(\Omega^2_X)$ is generated by a global nondegenerate 2-form.*

The geometry of irreducible symplectic varieties is quite restrictive. For instance, we have:

Theorem 4.2.10. *[Huy97] For X a K3 surface and D a divisor class of minimal degree, $M(r, D, a)$ is deformation equivalent to $X^{[g-ra]}$, where $X^{[n]}$ is the Hilbert scheme of n points.*

4.3 A Stratification of the Moduli Spaces

In the setup of (4.2.8) suppose further that $M(v)$ is a fine moduli space, so there exists a universal sheaf $\mathcal{F}$ on $X \times M(v)$, flat over $M(v)$, such that for every point $p = [\mathcal{E}] \in M(v)$, the restriction of $\mathcal{F}$ to $X \times p$ is $\mathcal{E}$. Let $\pi : X \times M(v) \to M(v)$ be the projection, and consider the subsets

$$M(v)_i = \{[\mathcal{E}] \in M(v) | \dim H^0(\mathcal{E}) = i\} \tag{4.4}$$

with the induced reduced subscheme structure. By the semicontinuity theorem, we have immediately

Lemma 4.3.1. $\{M(v)_i\}_{i\geq 0}$ *is a locally closed stratification of* $M(v)$*.*

In general $M(v)$ need not have a universal family, but étale locally it does, *cf.* (2.4.6). The cohomology of coherent sheaves can be computed étale locally, and closed and open immersions are both étale local properties, so

Proposition 4.3.2. $\{M(v)_i\}_{i\geq 0}$ *is a (finite) locally closed stratification of* $M(v)$*.*

The finiteness simply follows from the coherence of $\pi_*\mathcal{F}$ étale-locally. Since the second cohomology vanishes for any sheaf $\mathcal{E}$ with Mukai vector v,

$$\dim H^0(\mathcal{E}) \geq \chi(\mathcal{E}) = (v(\mathcal{O}), v) = r + a$$

The generic stratum is in fact $M(v)_{r+a}$.

4.4 Properties of Stable Sheaves on K3 Surfaces

Let X be a K3 surface over k with ample class H. Fix a divisor class D of minimal degree on X, and for $\mathcal{E}$ on X with $\mathrm{c}_1(\mathcal{E}) \in \mathbb{Z}D$, let $d = d(\mathcal{E})$ be the integer such that $c_1(\mathcal{E}) = dD$. We will be concerned throughout with sheaves $\mathcal{E}$ with $c_1(\mathcal{E}) = D$, so by (4.2.5) and (4.2.6), Gieseker stability for 2-dimensional sheaves is equivalent to μ-stability, and for 1-dimensional sheaves to purity. By stability henceforth we will mean Gieseker stability, but we will freely use the μ-stability criterion when it applies. This section is adapted from the treatment in [Yos99].

The following simple observation will give us surprising mileage in describing μ-stable sheaves whose first Chern class is D.

Lemma 4.4.1. *Suppose given integers $r, r_1, r_2, d, d_1, d_2, e$ with $r, r_1, r_2 \geq 0$ and $e > 0$ such that*

$$r_1 d - r d_1 \geq e \qquad r_1 d_2 - r_2 d_1 = e \qquad r d_2 - r_2 d \geq e \tag{4.5}$$

Then $r \geq r_1 + r_2$.

Proof.

$$re = r r_1 d_2 - r r_2 d_1 = r_1(r d_2 - r_2 d) + r_2(r_1 d - r d_1) \geq (r_1 + r_2)e$$

and therefore $r \geq r_1 + r_2$. □

We'll call two pairs of integers r_1, d_1 and r, d with $r_1, r \geq 0$ *adjacent* if $dr_1 - d_1 r = 1$. Since

$$1 = dr_1 - d_1 r = (dr_1 - d_1 r) - (dr_1 - d_1 r)$$

if r_1, d_1 and r, d are adjacent, so are r_1, d_1 and $r_2 = r - r_2, d_2 = d - d_2$. We'll call two sheaves $\mathcal{E}_1$ and $\mathcal{E}$ on X *D-adjacent* if $c_1(\mathcal{E}_1), c_1(\mathcal{E}) \in \mathbb{Z}D$ and $d(\mathcal{E}_1), \mathrm{rk}(\mathcal{E}_1)$ and $d(\mathcal{E}), \mathrm{rk}(\mathcal{E})$ are adjacent. Thus, for any short exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

$\mathcal{E}_1, \mathcal{E}$ are D-adjacent if and only if $\mathcal{E}, \mathcal{E}_2$ are. (4.4.1) has the following consequence

Corollary 4.4.2. *Let D be a divisor class of minimal degree on X, and $\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2$ sheaves on X with $\mathcal{E}_1, \mathcal{E}_2$ D-adjacent. If*

$$\mu(\mathcal{E}_1) < \mu(\mathcal{E}) < \mu(\mathcal{E}_2)$$

then $\mathrm{rk}(\mathcal{E}) \geq \mathrm{rk}(\mathcal{E}_1) + \mathrm{rk}(\mathcal{E}_2)$.

Proof. Since $\mu(\mathcal{E}_1) < \mu(\mathcal{E})$, we must have $\mathrm{rk}(\mathcal{E}) \deg \mathcal{E}_1 < \mathrm{rk}(\mathcal{E}_1) \deg \mathcal{E}$; D is of minimal degree, so the difference must be at least $D.H$. Similarly, we have $\mathrm{rk}(\mathcal{E}) \deg \mathcal{E}_2 - \mathrm{rk}(\mathcal{E}_2) \deg \mathcal{E} \geq D.H$. Applying the lemma with $d_i = \deg \mathcal{E}_i$, $r_i = \mathrm{rk}(\mathcal{E}_i)$ for $i \in \{\varnothing, 1, 2\}$ and $e = D.H$, the result follows. □

It is almost the case that morphisms between D-adjacent stable sheaves are either surjective or injective; this is reminiscent of (2.3). Recall that a morphism of sheaves $\varphi : \mathcal{E} \to \mathcal{F}$ is a surjection (injection, isomorphism) in codimension 1 if the cokernel (kernel, cokernel and kernel) is supported in codimension 2—that is, if the image of φ in $\mathrm{Coh}_{0,2}(X)$ is a surjection (injection, isomorphism). Obviously if $\mathcal{F} \in \mathrm{Ob}(\mathrm{Coh}(X))$ then such a φ is injective in codimension 1 if and only if it is injective.

Lemma 4.4.3. *Assume $\mathcal{E}_1, \mathcal{E}$ are stable and D-adjacent with $\mathcal{E}_1$ is locally free; let $U \subset \mathrm{Hom}(\mathcal{E}_1, \mathcal{E})$ be a subspace. Then the evaluation morphism $\varphi : U \otimes \mathcal{E}_1 \to \mathcal{E}$ is either injective or surjective in codimension 1. Further, if φ is injective,* $\operatorname{coker} \varphi$ *is stable.*

Proof. [Yos99] Suppose first that $\dim U = 1$, so $\varphi : \mathcal{E}_1 \to \mathcal{E}$ is a nonzero morphism. Also assume that $\mathrm{rk}(\mathcal{E}) > 0$. Let $\mathcal{G}$ be the image of φ, so φ factors as

$$\mathcal{E}_1 \twoheadrightarrow \mathcal{G} \hookrightarrow \mathcal{E} \tag{4.6}$$

By (4.4.2), one of the two morphisms in (4.6) is trivial in $\mathrm{Coh}_{0,2}(X)$, so φ is either surjective in codimension 1 or injective, by 2.3.*iii*. If $\mathrm{rk}(\mathcal{E}) = 0$, then $\mathcal{E}$ is pure and rank one on it's support, which by (4.2.3) is both reduced and irreducible. Thus, φ must be surjective in codimension 1.

If φ is injective, let $\mathcal{E}_2$ be the cokernel of φ, so there is an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

If $\mathcal{E}_2$ is not stable and $\mathrm{rk}(\mathcal{E}_2) > 0$, there is a maximal destabilizing quotient $\mathcal{E}_2 \twoheadrightarrow \mathcal{G}$ in $\mathrm{Coh}_{0,2}(X)$ such that

$$\mu(\mathcal{E}) < \mu(\mathcal{G}) < \mu(\mathcal{E}_2)$$

which is impossible by (4.4.2), since we must have $\mathrm{rk}(\mathcal{E}) > 0$. If $\mathrm{rk}(\mathcal{E}_2) = 0$, we need only show that $\mathcal{E}_2$ is pure, but this follows from the Serre criterion [HL] that $\mathcal{E}xt^2(\mathcal{E}_2, \mathcal{O}) = 0$, since $\mathcal{E}xt^2(\mathcal{E}, \mathcal{O}) = 0$.

Now assume U general. Choose a splitting $U \cong V \oplus W$ with W 1-dimensional. There is a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & U \otimes \mathcal{E}_1 & \longrightarrow & V \otimes \mathcal{E}_1 & \longrightarrow & 0 \\ & & \downarrow \psi & & \downarrow \varphi & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E} & = & \mathcal{E} & \longrightarrow & 0 & & \end{array}$$

By the snake lemma, there's an exact sequence

$$0 \to \ker \psi \to \ker \varphi \to V \otimes \mathcal{E}_1 \to \operatorname{coker} \psi \to \operatorname{coker} \varphi \to 0$$

Proceeding inductively, if φ is not surjective in codimension 1, then neither is ψ, and therefore ψ is injective. $\operatorname{coker} \psi$ is then stable, and the boundary map $V \otimes \mathcal{E}_1 \to \operatorname{coker} \psi$ is injective. □

Thus, when $U \otimes \mathcal{E}_1 \to \mathcal{E}$ is injective, the cokernel is stable. Dually, when $U \otimes \mathcal{E}_1 \to \mathcal{E}$ is surjective in codimension 1, the kernel is stable, though we will not need this:

Lemma 4.4.4. *Let $\mathcal{E}_1, \mathcal{E}$ be stable and D-adjacent with $\mathcal{E}_1$ locally free. Let $U \subset \operatorname{Hom}(\mathcal{E}_1, \mathcal{E})$. If evaluation map $U \otimes \mathcal{E}_1 \to \mathcal{E}$ is surjective in codimension 1, then the kernel is stable.*

Proof. [Yos99] Let $n = \dim U$, and $\mathcal{F} = \ker \varphi$; $\mathcal{F}$ is automatically either 0 or 2-dimensional. Assuming $\mathcal{F}$ is not stable, there is a maximal destabilizing subsheaf $\mathcal{G}$ in $\operatorname{Coh}_{0,2}(X)$ with

$$\mu(\mathcal{F}) = \frac{nd_1 - d}{nr_1 - r} < \mu(\mathcal{G})$$

$U \otimes \mathcal{E}_1$ is semistable of slope $\mu(\mathcal{E}_1)$, and $\mathcal{G}$ is a subsheaf, so in fact

$$\frac{nd_1 - d}{nr_1 - r} < \mu(\mathcal{G}) \leq \mu(\mathcal{E}_1)$$

and by (4.4.2) the right inequality is equality, since $\mathcal{G}$ must have strictly smaller rank than $\mathcal{F}$. For any one-dimensional quotient $U \to V$, the composition $\mathcal{G} \to V \otimes \mathcal{E}_1 \cong \mathcal{E}_1$ is trivial in $\operatorname{Coh}_{0,2}(X)$ by (2.3.*iii*), so $\mathcal{G} \cong \mathcal{E}_1$ in codimension 1, and therefore $\operatorname{Hom}(\mathcal{E}_1, \mathcal{F}) \neq 0$, a contradiction since $U \subset \operatorname{Hom}(\mathcal{E}_1, \mathcal{E})$. □

Finally, any extension of D-adjacent stable sheaves is stable:

Lemma 4.4.5. *Let $\mathcal{E}_1, \mathcal{E}_2$ be D-adjacent stable sheaves with $\mathcal{E}_1$ locally free. For any nontrivial extension nonzero subspace $V \subset \operatorname{Ext}^1(\mathcal{E}_2, \mathcal{E}_1)$, the corresponding extension*

$$0 \to V^* \otimes \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0 \tag{4.7}$$

defines a stable sheaf $\mathcal{E}$.

Proof. [Yos99] Assume first that $\dim V = 1$, so there is an extension

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

As noted above, $\mathcal{E}_1, \mathcal{E}$ are also D-adjacent. Let $r_i = \mathrm{rk}(\mathcal{E}_i)$ for $i \in \{\varnothing, 1, 2\}$, and assume that $r_1 < r$. If $\mathcal{E}$ is not stable, there is a maximal destabilizing sheaf $\mathcal{G} \hookrightarrow \mathcal{E}$ in $\mathrm{Coh}(X)$ with $\mu(\mathcal{G}) > \mu(\mathcal{E})$. Thus,

$$\mu(\mathcal{E}_1) < \mu(\mathcal{E}) < \mu(\mathcal{G})$$

$\mathcal{E}_1$ is stable, so $\mathcal{G} \hookrightarrow \mathcal{E}$ cannot factor through $\mathcal{E}_1 \hookrightarrow \mathcal{E}$, and the map $\mathcal{G} \to \mathcal{E} \to \mathcal{E}_2$ is nonzero. $\mathcal{E}_2$ is stable, so $\mu(\mathcal{G}) \leq \mu(\mathcal{E}_2)$. We cannot have $\mathrm{rk}(\mathcal{G}) \geq r + r_2$, so by (4.4.2) there is actually equality $\mu(\mathcal{G}) = \mu(\mathcal{E}_2)$. Thus, by (2.3.*iii*) $\mathcal{G} \to \mathcal{E}_2$ is injective and surjective in codimension 1. $\mathcal{E}_1$ is locally free, so $\mathcal{H}om(\mathcal{E}_2/\mathcal{G}, \mathcal{E}_1) = 0$.

Let $e \in \mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_2)$ be the extension class of (4.7). e is the image of the identity element of $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_1)$ under the left map in the exact sequence

$$\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_1) \to \mathrm{Ext}^1(\mathcal{E}_2, \mathcal{E}_1) \to \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}_1)$$

There is also an exact sequence

$$\mathrm{Ext}^1(\mathcal{E}_2/\mathcal{G}, \mathcal{E}_1) \to \mathrm{Ext}^1(\mathcal{E}_2, \mathcal{E}_1) \to \mathrm{Ext}^1(\mathcal{G}, \mathcal{E}_1) \tag{4.8}$$

The second map factors as $\mathrm{Ext}^1(\mathcal{E}_2, \mathcal{E}_1) \to \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}_1) \to \mathrm{Ext}^1(\mathcal{G}, \mathcal{E}_1)$, and therefore maps e to 0. $\mathcal{E}_1$ is locally free, so $\mathcal{E}xt^1(\mathcal{E}_2/\mathcal{G}, \mathcal{E}_1) = 0$. By the local-to-global spectral sequence, $\mathrm{Ext}^1(\mathcal{E}_2/\mathcal{G}, \mathcal{E}_1) = 0$, and therefore the right map in (4.8) is injective, and $e = 0$ which is a contradiction. Thus $\mathcal{E}$ is stable.

Now suppose $r_1 = r$, in which case $r = r_1 = 1$ and $d = d_2 = 1$. We need only show $\mathcal{E}$ is torsion-free. If $\mathcal{G} \neq 0$ is the torsion subsheaf of $\mathcal{E}$, then $\mathcal{G} \hookrightarrow \mathcal{E}_2$, and since $\mathcal{E}_2$ is pure of dimension 1, $\mathcal{E}_2/\mathcal{G}$ is 0-dimensional. Again we have $\mathrm{Ext}^1(\mathcal{E}_2/\mathcal{G}, \mathcal{E}_1) = 0$, and the extension is trivial by the same argument.

For general V, choose a quotient $V \to V'$ with $\dim V' = \dim V - 1$. There is a diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & V' \otimes \mathcal{E}_1 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{E}_2 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & V \otimes \mathcal{E}_1 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}_2 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{E}_1 & = & \mathcal{E}_1 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

With exact rows and columns. By induction, $\mathcal{E}'$ is stable and therefore $\mathcal{E}$ is. $\square$

4.5 Stable Pairs on K3 surfaces.

We will apply the results of the last section in the special case $\mathcal{E}_1 = \mathcal{O}$. Given a sheaf $\mathcal{E}$, $\mathcal{O}$ and $\mathcal{E}$ are D-adjacent if $\mathrm{c}_1(\mathcal{E}) \in \mathbb{Z}D$ and $d(\mathcal{E}) = 1$—that is, $\mathrm{c}_1(\mathcal{E}) = D$. The results of the last section can be summarized:

Proposition 4.5.1. *Let X be a K3 surface and D a divisor class on X of minimal degree. Given a level n stable pair $\varphi : U \otimes \mathcal{O} \to \mathcal{E}$ with $\mathrm{c}_1(\mathcal{E}) = D$, φ is injective if $n \leq \mathrm{rk}(\mathcal{E})$. Further, for a vector space V over k and any extension*

$$0 \to V \otimes \mathcal{O} \to \mathcal{E} \to \mathcal{F} \to 0$$

$\mathcal{E}$ is stable with $\mathrm{c}_1(\mathcal{E}) = D$ if and only if $\mathcal{F}$ is stable with $\mathrm{c}_1(\mathcal{F}) = D$.

These properties together strongly restrict the geometry of the moduli spaces of stable pairs on K3 surfaces:

Theorem 4.5.2. *[KY00] Let X be a K3 surface, $v = (v_0, v_1, v_2) \in H^*(X)$ a Mukai vector. For $v_1 = D$ of minimal degree, $\mathrm{Syst}^n(v)$ is smooth.*

Proof. Given a stable pair $\Lambda = (\mathcal{E}, U)$ on X, let $x \in \mathrm{Ob}(D^b(X))$ be the class of the complex

$$x = [U \otimes \mathcal{O} \to \mathcal{E}] \tag{4.9}$$

with $\mathcal{E}$ placed in degree 1, as in Section 3.3. Again, there is a triangle

$$x \to U \otimes \mathcal{O} \to \mathcal{E} \to x[1] \tag{4.10}$$

Taking $\mathrm{RHom}(\,\cdot\,, \mathcal{E})$ gives a long exact sequence

$$\begin{array}{ccccc} & & 0 & \longrightarrow & \mathrm{Ext}^{-1}(x,\mathcal{E}) \\ \longrightarrow \mathrm{Hom}(\mathcal{E},\mathcal{E}) & \longrightarrow & \mathrm{Hom}(U\otimes\mathcal{O},\mathcal{E}) & \longrightarrow & \mathrm{Hom}(x,\mathcal{E}) \\ \longrightarrow \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) & \longrightarrow & \mathrm{Ext}^1(U\otimes\mathcal{O},\mathcal{E}) & \longrightarrow & \mathrm{Ext}^1(x,\mathcal{E}) \\ \longrightarrow \mathrm{Ext}^2(\mathcal{E},\mathcal{E}) & \longrightarrow & \mathrm{Ext}^2(U\otimes\mathcal{O},\mathcal{E}) = 0 & & \end{array}$$

where $0 = \mathrm{Ext}^2(U \otimes \mathcal{O}, \mathcal{E}) = U \otimes H^2(\mathcal{E})$ since $\mathcal{E}$ is stable. By Section 3.3, the tangent space to $\mathrm{Syst}^n(v)$ at Λ is $\mathrm{Hom}(x, \mathcal{E})$ and the obstruction lies in the kernel of the composition

$$\mathrm{Ext}^1(x, \mathcal{E}) \to \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) \xrightarrow{\mathrm{tr}} H^2(\mathcal{O}) \tag{4.11}$$

whose Serre dual is

$$H^0(\mathcal{O}) \xrightarrow{\mathrm{id}} \mathrm{Hom}(\mathcal{E}, \mathcal{E}) \hookrightarrow \mathrm{Ext}^1(\mathcal{E}, x) \tag{4.12}$$

$\mathcal{E}$ is stable and therefore simple, so $\mathrm{Hom}(\mathcal{E}, \mathcal{E})$ is one-dimensional. If we show $\mathrm{Ext}^1(\mathcal{E}, x)$ is also one-dimensional, the obstruction space will vanish and the result will follow. Following [KY00], let $\mathcal{G}$ be the universal extension

$$0 \to \mathrm{Ext}^1(\mathcal{E}, \mathcal{O})^* \otimes \mathcal{O} \to \mathcal{G} \to \mathcal{E} \to 0 \tag{4.13}$$

corresponding to the identity subspace of $\mathrm{Ext}^1(\mathcal{E},\mathcal{O})$. The composition in (4.12) is injective so it suffices to show that

(a) The composition map $\mathrm{Ext}^1(\mathcal{E},x) \to \mathrm{Ext}^1(\mathcal{G},x)$ is injective

(b) $\mathrm{Ext}^1(\mathcal{G},x)$ is one-dimensional

For then $\mathrm{Ext}^1(\mathcal{E},x)$ is one-dimensional, (4.12) is surjective, and the kernel of (4.11) is trivial. Applying $R\,\mathrm{Hom}(\,\cdot\,,x)$ to (4.13), we have an exact sequence of the form

$$\mathrm{Ext}^1(\mathcal{E},\mathcal{O})^* \otimes \mathrm{Hom}(\mathcal{O},x) \to \mathrm{Ext}^1(\mathcal{E},x) \to \mathrm{Ext}^1(\mathcal{G},x) \tag{4.14}$$

The long exact (global) cohomology sequence associated to (4.10) begins with

$$0 \to H^0(x) \to H^0(U \otimes \mathcal{O}) \to H^0(\mathcal{E})$$

and by the stability of Λ the rightmost map is injective, so

$$0 = H^0(x) = \mathrm{Hom}(\mathcal{O},x)$$

and (4.14) yields (a).

The long exact sequence associated to (4.13) yields

$$0 = \mathrm{Ext}^1(\mathcal{E},\mathcal{O})^* \otimes H^1(\mathcal{O}) \to H^1(\mathcal{G}) \to H^1(\mathcal{E}) \to \mathrm{Ext}^1(\mathcal{E},\mathcal{O})^* \otimes H^2(\mathcal{O}) \to 0$$

Of course, by Serre duality $\mathrm{Ext}^1(\mathcal{E},\mathcal{O})^* \cong H^1(\mathcal{E})$, so $H^1(\mathcal{G}) = 0$, and by Serre duality again $\mathrm{Ext}^1(\mathcal{G}, U \otimes \mathcal{O}) = 0$. By (4.5.1), $\mathcal{G}$ is stable, so $H^2(\mathcal{G}) = 0$, and thus using Serre duality again $\mathrm{Hom}(\mathcal{G}, U \otimes \mathcal{O}) = 0$. Applying $\mathrm{RHom}(G,\,\cdot\,)$ to (4.10) gives an exact sequence

$$0 = \mathrm{Hom}(\mathcal{G}, U \otimes \mathcal{O}) \to \mathrm{Hom}(\mathcal{G},\mathcal{E}) \to \mathrm{Ext}^1(\mathcal{G},x) \to \mathrm{Ext}^1(\mathcal{G}, U \otimes \mathcal{O}) = 0$$

and $\mathrm{Hom}(\mathcal{G},\mathcal{E}) \cong \mathrm{Ext}^1(\mathcal{G},x)$. But applying $\mathrm{RHom}(\mathcal{G},\,\cdot\,)$ to (4.13) gives one final an exact sequence

$$0 = \mathrm{Hom}(\mathcal{G},\mathcal{O}^d) \to \mathrm{Hom}(\mathcal{G},\mathcal{G}) \to \mathrm{Hom}(\mathcal{G},\mathcal{E}) \to \mathrm{Ext}^1(\mathcal{G},\mathcal{O}^d) = 0$$

and since $\mathcal{G}$ is simple, (b) follows. □

Recall from Section 4.3 that

$$M(v)_i = \{\mathcal{E} \in M(v) | h^0(\mathcal{E}) = i\}$$

form a locally closed stratification of $M(v)$. Denote by $\mathrm{Syst}^n(v)_i$ the preimage of $M(v)_i$ under the forgetful morphism $p : \mathrm{Syst}^n(v) \to M(v)$; clearly $\{\mathrm{Syst}^n(v)\}_{i\geq 0}$ is a locally closed stratification of $\mathrm{Syst}^n(v)$.

For $v = (r, D, a)$, denote $\mathrm{Syst}^n(r, D, a) = \mathrm{Syst}^n(v)$ and $M(r, D, a) = M(v)$. For $r \geq n$ there is a map (*cf.* [KY00]) $q : \mathrm{Syst}^n(r, D, a) \to M(r-n, D, a-n)$ mapping $(\mathcal{E}, U)$ to the cokernel $\mathcal{F}$ of the structure map

$$0 \to U \otimes \mathcal{O} \to \mathcal{E} \to \mathcal{F} \to 0$$

By (4.5.1) $\mathcal{F}$ is stable, and obviously $v(\mathcal{F}) = v(\mathcal{E}) - v(\mathcal{O}^n) = (r-n, D, a-n)$ since $v(\mathcal{O}) = (1, 0, 1)$. Further, since $H^1(U \otimes \mathcal{O}) = 0$, the stratum $\mathrm{Syst}^n(r, D, a)_i$ maps into $M(r-n, D, a-n)_{i-n}$.

Theorem 4.5.3 ([KY00]). *1. The restriction* $\mathrm{Syst}^n(v)_i \to M(v)_i$ *of the forgetful morphism p is an étale-locally trivial fibration with fiber* $\mathrm{Gr}(n, i)$.

2. The restriction $\mathrm{Syst}^n(r, D, a)_i \to M(r-n, D, a-n)_{i-n}$ *of the quotient morphism q is an étale-locally trivial fibration with fiber* $\mathrm{Gr}(n, n+i-r-a)$.

Proof. 1. Assume first that $M(v)$ has a universal sheaf $\mathcal{F}$ on $X \times M(v)$, and let

$\pi : X \times M(v) \to M(v)$ be the projection. $\mathrm{Syst}^n(v) \xrightarrow{p} M(v)$ can be explicitly constructed as a relative Grassmannian of n-planes in $\pi_*\mathcal{F}$. $R\pi_*\mathcal{F}$ is supported in cohomological degrees 0 and 1, since $H^2(X_p, \mathcal{F}_p) = 0$ for each fiber X_p by stability. By cohomology and base change, the restriction of $R^1\pi_*\mathcal{F}$ to $M(v)_i$ is locally free since the groups $H^1(X_p, \mathcal{F}_p)$ for all $p \in M(v)$ have constant dimension $i - \chi(v) = i - r - a$. By the flatness of $\mathcal{F}$ over $M(v)$, $\pi_*\mathcal{F}$ is locally free of rank i, and $\mathrm{Syst}^n(v)_i$ is a Zariski-locally trivial fibration with fiber $\mathrm{Gr}(n,i)$ over $M(v)_i$. $M(v)$ étale locally has a universal sheaf, and it follows that $\mathrm{Syst}^n(v)_i \to M(v)_i$ is an étale-locally trivial $\mathrm{Gr}(n,i)$ fibration.

2. The proof is exactly analogous to the proof of the first part, except now $\mathrm{Syst}^n(r,D,a) \xrightarrow{q} M(r-n, D, a-n)$ is constructed as the relative Grassmannian of n-planes in $R^1\pi_*\mathcal{F}$, since every extension yields a stable pair, by (4.5.1). $R^1\pi_*\mathcal{F}$ is locally free over $M(r-n,D,a-n)_{i-n}$; the fiber over $[\mathcal{E}] \in M(r-n,D,a-n)_{i-n}$ is $H^1(\mathcal{E})$ and therefore its rank is

$$i - n - \chi(\mathcal{E}) = i - n - (r + a - 2n) = n + i - r - a$$

□

The main tool for the computation of the Hodge polynomials of $\mathrm{Syst}^n(r,D,a)$ will be the existence of the diagrams

$$\begin{array}{ccc} & \mathrm{Syst}^n(r,D,a)_i & \\ \overset{p}{\swarrow} & & \overset{q}{\searrow} \\ M(r,D,a)_i & & M(r-n,D,a-n)_{i-n} \end{array}$$

where p is an étale-local $\mathrm{Gr}(n,i)$-fibration and q is an étale local $\mathrm{Gr}(n, n+i-r-a)$-fibration.

One final property of the stable pair moduli spaces that will be relevant later is the duality

Theorem 4.5.4. *[KY00] In the setup of* (4.5.2) *there is an isomorphism*

$$\mathrm{Syst}^n(r, D, a) \cong \mathrm{Syst}^n(n - r, D, a - r)$$

for all $r \leq n$.

Proof. We will at the very least define the map; see [KY00] for the proof of the theorem. Let $U \otimes \mathcal{O} \to \mathcal{E}$ be a stable pair, and let $x \in D^b(X)$ be the cone as in (4.9). Applying $R\mathcal{H}om(\,\cdot\,, \mathcal{O})$ to the triangle (4.10), we have

$$U^* \otimes \mathcal{O} \cong \mathcal{H}om(U \otimes \mathcal{O}, \mathcal{O}) \to \mathcal{H}om(x, \mathcal{E}) \xrightarrow{\cong} \mathcal{E}xt^1(\mathcal{E}, \mathcal{O})$$

One can show that $U^* \otimes \mathcal{O} \to \mathcal{E}xt^1(\mathcal{E}, \mathcal{O})$ is a stable pair and that this defines the isomorphism. □

Chapter 5

u-Calculus

The computation of chapter 6 is best expressed in terms of "u"-calculus. If we were to follow convention, this section would be called "q-calculus," since usually the formal variable used is "q." q will have a different use for us later, and so we'll use u instead.

5.1 u-Binomial Coefficients

The u-integer $[n]$ is the polynomial in u given by

$$[n] = \frac{u^n - 1}{u - 1}$$

The u-factorial and u-binomial coefficients are defined similarly:

$$[n]! = \prod_{s=1}^{n} [s] \qquad\qquad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$$

For $k \leq n$ and 0 for $k > n$; by fiat $[0]! = 1$. Given f an element of a reasonable ring of power series in u, x (for example Laurent series), the u-derivative is

$$\left(\frac{d}{dx}\right)_u f = \frac{f(ux) - f(x)}{ux - x}$$

For example, we have

$$\left(\frac{d}{dx}\right)_u (x^n) = \frac{u^n x^n - x^n}{ux - x} = [n]x^{n-1}$$

5.2 Properties of u-Binomial Coefficients

Most binomial identities have u-analogs, many of which recover the classical identities in the $u \to 1$ limit:

Lemma 5.2.1. *For any $k \leq n$*

1.

$$[n] = [n-k] + u^{n-k}[k]$$

2.

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} + u^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix} \tag{5.1}$$

Proof. 1. Follows immediately from $[n+1] = \sum_{s=0}^{n} u^s$.

2.

$$\begin{aligned}
\begin{bmatrix} n+1 \\ k \end{bmatrix} &= \frac{[n+1]!}{[k]![n+1-k]!} \\
&= \frac{[n]!}{[k]![n-k]!}\left(\frac{[n+1]}{[n+1-k]}\right) \\
&= \frac{[n]!}{[k]![n-k]!}\left(1+u^{n+1-k}\frac{[k]}{[n+1-k]}\right) \\
&= \begin{bmatrix} n \\ k \end{bmatrix} + u^{n+1-k}\begin{bmatrix} n \\ k-1 \end{bmatrix}
\end{aligned}$$

□

Note that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ has degree $k(n-k)$. The symmetric u-binomial coefficient is defined for $0 \leq k \leq n$ by

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = u^{-\frac{k(n-k)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}$$

Also, under the same conditions let

$$\begin{Bmatrix} -n \\ k \end{Bmatrix} = (-1)^k \begin{Bmatrix} n+k-1 \\ k \end{Bmatrix}$$

Let

$$K_n(t,u) = \prod_{s=0}^{n-1}(1+tu^{s-\frac{n-1}{2}})$$

for $n \geq 0$.

Lemma 5.2.2.

$$K_n(t^{-1},u) = t^{-n}K_n(t,u)$$

Proof.

$$K_n(t^{-1},u) = t^{-n}\prod_{s=0}^{n-1}(t+u^{s-\frac{n-1}{2}})$$

but terms in the product come in pairs $(t+u^s)(t+u^{-s}) = (1+tu^s)(1+tu^{-s})$. $\square$

K_n is invertible as a Laurent series in $t, u^{\frac{1}{2}}$; let

$$K_{-n}(t,u) = K_n(t,u)^{-1}$$

There is an analog of (5.2.1) for symmetric u-binomial coefficients:

Lemma 5.2.3. *For any* $0 \le k \le n$

1.

$$\begin{Bmatrix} n+1 \\ k \end{Bmatrix} = u^{-\frac{k}{2}} \begin{Bmatrix} n \\ k \end{Bmatrix} + u^{\frac{n+1-k}{2}} \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \tag{5.2}$$

2. $K_n(t,u)$ *is the generating function for the* $\begin{Bmatrix} n \\ k \end{Bmatrix}$, *that is*

$$K_n(t,u) = \sum_{k=0}^{\infty} t^k \begin{Bmatrix} n \\ k \end{Bmatrix}$$

3.

$$\begin{Bmatrix} n+k \\ k \end{Bmatrix} = \sum_{s=0}^{k} u^{\frac{sn+s-k}{2}} \begin{Bmatrix} n+k-s-1 \\ k-s \end{Bmatrix}$$

4. $K_{-n}(t,u)$ *is the generating function for the* $\begin{Bmatrix} -n \\ k \end{Bmatrix}$, *that is*

$$K_{-n}(t,u) = \sum_{k=0}^{\infty} t^k \begin{Bmatrix} -n \\ k \end{Bmatrix}$$

Proof. 1. Multiplying (5.1) by $u^{\frac{k(n+1-k)}{2}}$ gives (5.2).

2. Note that

$$K_{n+1}(t,u) = \left(1 + tu^{\frac{n}{2}}\right) K_n(tu^{-\frac{1}{2}}, u) \tag{5.3}$$

Assuming by induction that the coefficient of t^s in $K_n(tu^{-\frac{1}{2}}, u)$ is $u^{-\frac{s}{2}}\begin{Bmatrix} n \\ s \end{Bmatrix}$, the coefficient of t^k in $K_{n+1}(t,u)$ is

$$u^{-\frac{k}{2}}\begin{Bmatrix} n \\ k \end{Bmatrix} + u^{\frac{n-k+1}{2}}\begin{Bmatrix} n \\ k-1 \end{Bmatrix}$$

which yields the result given part (1).

3. Replacing n in (5.2) with $n+k-1$ we have

$$\begin{Bmatrix} n+k \\ k \end{Bmatrix} = u^{-\frac{k}{2}}\begin{Bmatrix} n+k-1 \\ k \end{Bmatrix} + u^{\frac{n}{2}}\begin{Bmatrix} n+k-1 \\ k-1 \end{Bmatrix} \tag{5.4}$$

Note that

$$\sum_{s=0}^{k} u^{\frac{sn+s-k}{2}}\begin{Bmatrix} n+k-s-1 \\ k-s \end{Bmatrix} = u^{-\frac{k}{2}}\begin{Bmatrix} n+k-1 \\ k \end{Bmatrix} + \sum_{s=1}^{k} u^{\frac{sn+s-k}{2}}\begin{Bmatrix} n+k-s-1 \\ k-s \end{Bmatrix}$$
$$= u^{-\frac{k}{2}}\begin{Bmatrix} n+k-1 \\ k \end{Bmatrix} + u^{\frac{n}{2}}\left(\sum_{s=0}^{k-1} u^{\frac{sn+s-k+1}{2}}\begin{Bmatrix} n+k-s-2 \\ k-s-1 \end{Bmatrix}\right)$$

By induction the term in parentheses is $\begin{Bmatrix} n+k-1 \\ k-1 \end{Bmatrix}$, and by (5.4) the result follows.

4. Inverting (5.3), we have

$$K_{-n-1}(t,u) = \frac{1}{1+tu^{\frac{n}{2}}}K_{-n}(tu^{-\frac{1}{2}}, u) = K_{-n}(tu^{-\frac{1}{2}}, u)\sum_{s=0}^{\infty}(-1)^s t^s u^{\frac{ns}{2}}$$

Inductively assuming the coefficient of t^{k-s} in $K_{-n}(tu^{-\frac{1}{2}}, u)$ is

$$u^{-\frac{k-s}{2}}\begin{Bmatrix} -n \\ s \end{Bmatrix} = (-1)^{k-s}u^{-\frac{k-s}{2}}\begin{Bmatrix} n+k-s-1 \\ k-s \end{Bmatrix}$$

the coefficient of t^k in $K_{-n-1}(t,u)$ is

$$(-1)^k \sum_{s=0}^{k} u^{\frac{ns+s-k}{2}} \begin{Bmatrix} n+k-s-1 \\ k-s \end{Bmatrix} = (-1)^k \begin{Bmatrix} n+k \\ k \end{Bmatrix} = \begin{Bmatrix} -n-1 \\ k \end{Bmatrix}$$

by part (3).

□

5.3 q-Theta Functions

Given expressions a, b polynomial in q, the Pochhammer symbol $(a,b)_\infty$ is a formal power series in q defined by

$$(a,b)_\infty = \prod_{n=0}^{\infty}(1-ab^n)$$

For example, $(q,q)_\infty = \prod_{n\geq 1}(1-q^n)$. The q-theta function $\Theta(x)$ is a formal power series in x, q defined by

$$\Theta(x) = (q,q)_\infty(x,q)_\infty(x^{-1}q,q)_\infty = (1-x)\prod_{n=1}^{\infty}(1-q^n)(1-xq^n)(1-x^{-1}q^n)$$

So in particular $\Theta(x)$ has a simple root at $x=1$. Our main use for $\Theta(x)$ is derived from the fact that

Lemma 5.3.1. *For $n \in \mathbb{Z}$, define*

$$\operatorname{sign}(n) = \begin{cases} +1 & n \geq 0 \\ -1 & n < 0 \end{cases}$$

Then for expressions a, b polynomial in q

$$\sum_{\text{sign}(i)=\text{sign}(j)} \text{sign}(i) a^i b^j q^{ij} = \frac{(q,q)_\infty^3 \Theta(ab)}{\Theta(a)\Theta(b)}$$

Proof. See [Hic88]. □

Define

$$\Psi(x,y) = \sum_{\ell \geq 0} \sum_{p \geq 1} (x^p - x^{-\ell}) y^{p-\ell} q^{p\ell}$$

Note that this can be rewritten[1]

$$\begin{aligned}\Psi(x,y) &= \left(\sum_{\ell \geq 0} \sum_{p \geq 1} (xy)^p y^{-\ell} + \sum_{\ell \geq 0} y^{-\ell} \right) - \left(\sum_{\ell \geq 0} \sum_{p \geq 1} (xy)^{-\ell} y^p - \sum_{p \geq 1} y^p \right) \\ &= \sum_{p,\ell \geq 0} (xy)^p (y^{-1})^\ell q^{p\ell} - \sum_{p,\ell \geq 1} (xy)^{-\ell} (y^{-1})^{-p} q^{p\ell}\end{aligned}$$

Corollary 5.3.2.

$$\Psi(x,y) = \frac{(q,q)_\infty^3 \Theta(x)}{\Theta(xy)\Theta(y^{-1})}$$

Explicitly,

$$\Psi(x,y) = \frac{(q,q)_\infty^2 (x,q)_\infty (x^{-1}q,q)_\infty}{(xyq,q)_\infty (x^{-1}y^{-1}q,q)_\infty (q,q)_\infty (q,q)_\infty}$$

[1] One must be careful about the ring of formal power series in which the identities below hold. We are using

$$\sum_{n \geq 0} y^{-n} + \sum_{n > 0} y^n = \frac{1}{1-y^{-1}} + \frac{y}{1-y} = 0$$

which must be justified delicately. See [Hic88, Zag91].

5.4 A Useful Matrix

In chapter 6 we will be interested in the matrix $\mathbf{A}(\mathbf{n}) = (A^n_{ij})_{i,j\geq 0}$ defined by

$$A^n_{ij} = \begin{cases} \begin{bmatrix} \frac{i+j}{2} \\ n \end{bmatrix} \begin{bmatrix} j \\ \frac{j-i}{2} \end{bmatrix} & i-j \equiv 0 \mod 2 \\ 0 & i-j \equiv 1 \mod 2 \end{cases}$$

i.e., the only nonzero entries are $A^n_{k,k+2\ell} = \begin{bmatrix} k+\ell \\ n \end{bmatrix} \begin{bmatrix} k+2\ell \\ \ell \end{bmatrix}$, $k, \ell \geq 0$. In particular, $A^0_{k,k+2\ell} = \begin{bmatrix} k+2\ell \\ \ell \end{bmatrix}$. $\mathbf{A}(\mathbf{0})$ is upper triangular with ones along the diagonal, and is therefore invertible:

Proposition 5.4.1. *The inverse of* $\mathbf{A}(\mathbf{0})$ *is the matrix* $\mathbf{B} = (B_{ij})_{i,j\geq 0}$ *given by*

$$B_{k,k+2\ell} = (-1)^l u^{\binom{\ell}{2}} \frac{[k+2\ell]}{[k+\ell]} \begin{bmatrix} k+\ell \\ \ell \end{bmatrix}$$

and $B_{k,k+2\ell+1} = 0$, *for* $k, \ell \geq 0$

Proof. We need only check that the $(k, k+2\ell)$ entry of $\mathbf{A}(\mathbf{0})\mathbf{B}$ for $\ell > 0$ is 0, since the diagonal terms are clearly 1 and both matrices are upper triangular. The relevant entries of $\mathbf{B}$ are

$$B_{k+2s,k+2\ell} = (-1)^{\ell-s} u^{\binom{\ell-s}{2}} \begin{bmatrix} k+\ell+s \\ \ell-s \end{bmatrix} \frac{[k+2\ell]}{[k+\ell+s]}$$

Also note that

$$\begin{aligned} \begin{bmatrix} k+2s \\ s \end{bmatrix} \begin{bmatrix} k+\ell+s \\ \ell-s \end{bmatrix} \frac{[k+2\ell]}{[k+\ell+s]} &= \left(\frac{[k+2s]!}{[s]![k+s]!} \right) \left(\frac{[k+s+\ell]!}{[\ell-s]![k+2s]!} \right) \frac{[k+2\ell]}{[k+\ell+s]} \\ &= \left(\frac{[\ell]!}{[s]![\ell-s]!} \right) \left(\frac{[k+s+\ell-1]!}{[k+s]![\ell-1]!} \right) \frac{[k+2\ell]}{[\ell]} \\ &= \begin{bmatrix} \ell \\ s \end{bmatrix} \begin{bmatrix} k+\ell-1 \\ \ell-1 \end{bmatrix} \frac{[k+2\ell]}{[\ell]} \end{aligned}$$

Thus

$$\begin{aligned}
\sum_{s=0}^{\infty} A^0_{k,k+2s} B_{k+2s,k+2\ell} &= \sum_{s=0}^{\ell} (-1)^{\ell-s} u^{\binom{\ell-s}{2}} \begin{bmatrix} k+2s \\ s \end{bmatrix} \begin{bmatrix} k+\ell+s \\ \ell-s \end{bmatrix} \frac{[k+2\ell]}{[k+\ell+s]} \\
&= \left(\frac{[k+2\ell]}{[\ell]} \right) \sum_{s=0}^{\ell} (-1)^{\ell-s} u^{\binom{\ell-s}{2}} \begin{bmatrix} k+s+\ell-1 \\ \ell-1 \end{bmatrix} \begin{bmatrix} \ell \\ s \end{bmatrix} \\
&= \left(\frac{[k+2\ell]}{[\ell]} \right) \sum_{s=0}^{\ell} (-1)^{\ell-s} u^{\binom{\ell-s}{2} + \frac{(\ell-1)(k+s)}{2} + \frac{s(\ell-s)}{2}} \begin{Bmatrix} k+s+\ell-1 \\ \ell-1 \end{Bmatrix} \begin{Bmatrix} \ell \\ s \end{Bmatrix} \\
&= u^{\frac{\ell^2-\ell+(\ell-1)k}{2}} \left(\frac{[k+2\ell]}{[\ell]} \right) \sum_{s=0}^{\ell} (-1)^{\ell-s} \begin{Bmatrix} k+s+\ell-1 \\ \ell-1 \end{Bmatrix} \begin{Bmatrix} \ell \\ s \end{Bmatrix} \\
&= (-1)^{k+\ell} u^{\frac{\ell^2-\ell+(\ell-1)k}{2}} \left(\frac{[k+2\ell]}{[\ell]} \right) \sum_{s=0}^{\ell} \begin{Bmatrix} -\ell \\ k+s \end{Bmatrix} \begin{Bmatrix} \ell \\ s \end{Bmatrix}
\end{aligned}$$

By (4) of (5.2.3), $\begin{Bmatrix} -\ell \\ k+s \end{Bmatrix}$ is the coefficient of t^{k+s} in $K_{-\ell}(t,q)$ and $\begin{Bmatrix} \ell \\ s \end{Bmatrix}$ is the coefficient of t^{-s} in $K_\ell(t^{-1},q)$. Therefore, the sum is the coefficient of t^k in $K_{-\ell}(t,q)K_\ell(t^{-1},q) = t^{-\ell}K_{-\ell}(t,q)K_\ell(t,q) = t^{-\ell}$ so it must be 0, unless $\ell = k = 0$, but we assumed $\ell > 0$. □

5.5 A Useful Product

As we shall see in (6.2.2), an explicit computation of the product $\mathbf{P}(\mathbf{n}) := \mathbf{A}(\mathbf{n})\mathbf{A}(\mathbf{0})^{-1}$ will enable us to perform the calculation in section 6.3. The product is a matrix $\mathbf{P}(\mathbf{n}) = (P^n_{ij})_{i,j\geq 0}$ given by

Lemma 5.5.1. *For $k, \ell \geq 0$, $n > 0$,*

$$P^n_{k,k+2\ell} = u^{\ell^2+\ell(k-n)} \frac{[k+2\ell]}{[n+\ell]} \begin{bmatrix} n+\ell \\ n \end{bmatrix} \begin{bmatrix} k+\ell-1 \\ n-1 \end{bmatrix}$$

and $P^n_{k,k+2\ell+1} = 0$.

Proof. The proof is a calculation very similar to the proof of lemma (5.2.3). Note that for $\ell \geq s$

$$\begin{aligned}
&\begin{bmatrix} k+s \\ n \end{bmatrix}\begin{bmatrix} k+2s \\ s \end{bmatrix}\begin{bmatrix} k+s+\ell \\ \ell-s \end{bmatrix} = \\
&\qquad = \frac{[k+s]\cdots[k+s-n+1]}{[n]!}\frac{[k+2s]\cdots[k+s+1]}{[s]!}\frac{[k+s+\ell]\cdots[k+2s+1]}{[\ell-s]!} \\
&\qquad = \frac{[k+s+\ell]!}{[n]![s]![\ell-s]![k+s-n]!} \\
&\qquad = \left(\frac{[n+\ell]!}{[n]![\ell]!}\right)\left(\frac{[\ell]!}{[s]![\ell-s]!}\right)\left(\frac{[k+s+\ell-1]!}{[k+s-n]![n+\ell-1]!}\right)\frac{[k+s+\ell]}{[n+\ell]}
\end{aligned}$$

so

$$\begin{aligned}
P^n_{k,k+2\ell} &= \sum_{s=0}^{\ell} A^n_{k,k+2s} B_{k+2s,k+2\ell} \\
&= \sum_{s=0}^{\ell} (-1)^{\ell-s} u^{\binom{\ell-s}{2}} \begin{bmatrix} k+s \\ n \end{bmatrix}\begin{bmatrix} k+2s \\ s \end{bmatrix}\begin{bmatrix} k+s+\ell \\ \ell-s \end{bmatrix}\frac{[k+2\ell]}{[k+s+\ell]} \\
&= \frac{[k+2\ell]}{[n+\ell]}\begin{bmatrix} n+\ell \\ n \end{bmatrix}\sum_{s=0}^{\ell}(-1)^{\ell-s}u^{\binom{\ell-s}{2}}\begin{bmatrix} k+s+\ell-1 \\ n+\ell-1 \end{bmatrix}\begin{bmatrix} \ell \\ s \end{bmatrix} \\
&= \frac{[k+2\ell]}{[n+\ell]}\begin{bmatrix} n+\ell \\ n \end{bmatrix}\sum_{s=0}^{\ell}(-1)^{\ell-s}u^{\binom{\ell-s}{2}+\frac{(n+\ell-1)(k-n+s)}{2}+\frac{s(\ell-s)}{2}}\begin{Bmatrix} k+s+\ell-1 \\ n+\ell-1 \end{Bmatrix}\begin{Bmatrix} \ell \\ s \end{Bmatrix} \\
&= \frac{[k+2\ell]}{[n+\ell]}\begin{bmatrix} n+\ell \\ n \end{bmatrix}u^{\frac{\ell^2-\ell+(n+\ell-1)(k-n)}{2}}\sum_{s=0}^{\ell}(-1)^{\ell-s}u^{sn/2}\begin{Bmatrix} k+s+\ell-1 \\ n+\ell-1 \end{Bmatrix}\begin{Bmatrix} \ell \\ s \end{Bmatrix} \\
&= (-1)^{k-n+\ell}\frac{[k+2\ell]}{[n+\ell]}\begin{bmatrix} n+\ell \\ n \end{bmatrix}u^{\frac{\ell^2-\ell+(n+\ell-1)(k-n)}{2}}\sum_{s=0}^{\ell}u^{sn/2}\begin{Bmatrix} -(n+\ell) \\ k-n+s \end{Bmatrix}\begin{Bmatrix} \ell \\ s \end{Bmatrix}
\end{aligned}$$

$u^{sn/2}\begin{Bmatrix} -(n+\ell) \\ k-n+s \end{Bmatrix}$ is the coefficient of t^{k-n+s} in $u^{(n^2-kn)/2}K_{-(n+\ell)}(tu^{n/2},u)$ and $\begin{Bmatrix} \ell \\ s \end{Bmatrix}$ is the coefficient of t^{-s} in $K_\ell(t^{-1},u)$. Therefore, the sum in (4.2.8) is the coefficient of t^{k-n} in

$$u^{(n^2-kn)/2}K_{-(n+\ell)}(tu^{n/2},u)K_\ell(t^{-1},u) = u^{(n^2-kn)/2}t^{-\ell}K_{-(n+\ell)}(tu^{n/2},u)K_\ell(t,u)$$
$$= u^{(n^2-kn)/2}t^{-\ell}K_{-n}(tu^{(n+\ell)/2},u)$$

which is

$$u^{\frac{\ell^2+\ell k}{2}}\left\{\begin{matrix} -n \\ k-n+\ell \end{matrix}\right\} = (-1)^{k-n+\ell}u^{\frac{\ell^2+\ell k}{2}}\left\{\begin{matrix} k+\ell-1 \\ n-1 \end{matrix}\right\}$$
$$= (-1)^{k-n+\ell}u^{\frac{\ell^2+\ell k-(n-1)(k+\ell-n)}{2}}\begin{bmatrix} k+\ell-1 \\ n-1 \end{bmatrix}$$

and we get

$$P^n_{k,k+2\ell} = u^{\ell^2+\ell k-n\ell}\frac{[k+2\ell]}{[n+\ell]}\begin{bmatrix} n+\ell \\ n \end{bmatrix}\begin{bmatrix} k+\ell-1 \\ n-1 \end{bmatrix}$$

□

The $n = 1$ case is of particular interest:

Corollary 5.5.2. *For $k, \ell \geq 0$*

$$P^1_{k,k+2\ell} = u^{\ell^2+\ell k-\ell}[k+2\ell]$$

Chapter 6

Computation of the Hodge Polynomials

6.1 Packaging the Results of Section 4.5

For X a scheme over k, let $\chi_{t,\bar{t}}(X) = \sum_{p,q\geq 0} h^{p,q}(X)(-t)^p(-\bar{t})^q$ denote the virtual Hodge polynomial of X. Throughout the following, we will set $u = t\bar{t}$ so that

$$\chi_{t,\bar{t}}(\mathbb{P}^n) = [n+1] := \frac{u^{n+1}-1}{u-1}$$

or more generally

$$\chi_{t,\bar{t}}(\mathrm{Gr}(k,n)) = \begin{bmatrix} n \\ k \end{bmatrix}$$

where $\mathrm{Gr}(k,n)$ is the Grassmannian of k planes in n-space.

Recall that for a divisor class $D \in H^2(X,\mathbb{Z})$, $D^2 = 2g-2$ by the adjunction formula, where g is the arithmetic genus of a divisor in the class D; g will be called the genus of D. For each genus $g \geq 0$ fix a polarized K3 surface X with a divisor class D of minimal degree and genus g, *cf.* (4.2.4)

- $g = 0, 1$: $X \to \mathbb{P}^1$ is an elliptic K3 with a section. $\mathrm{Pic}(X) = \mathbb{Z}\sigma \oplus \mathbb{Z}f$, where f

is the fiber class and σ the section class. For $g = 0$ take $H = \sigma + f$ and $D = \sigma$; for $g = 1$ take $H = \sigma + f$ and $D = f$.

- $g \geq 2$: X has Picard rank 1 with ample generator H of genus g ($H^2 = 2g - 2$); take $D = H$

Denote by $M(r, g, k)$ is the moduli space of H-stable rank r sheaves $\mathcal{E}$ with $\mathrm{c}_1(E) = D$ and $\mathrm{ch}_2(E).[X] = k$—that is, $v(\mathcal{E}) = (r, D, k)$. Define infinite matrices $\mathbf{M(g)} = (M(g)_{ij})_{i,j \geq 0}$ and $\mathbf{Syst^n(g)} = (\mathrm{Syst}^n(g)_{ij})_{i,j \geq 0}$ of Hodge polynomials by

$$M(g)_{ij} = \begin{cases} \chi_{t,\bar{t}}(M(\frac{i-j}{2}, g, \frac{i+j}{2})) & i - j \equiv 0 \mod 2 \\ 0 & i - j \equiv 1 \mod 2 \end{cases}$$

$$\mathrm{Syst}^n(g)_{ij} = \begin{cases} \chi_{t,\bar{t}}(\mathrm{Syst}^n(\frac{i-j}{2}, g, \frac{i+j}{2})) & i - j \equiv 0 \mod 2 \\ 0 & i - j \equiv 1 \mod 2 \end{cases}$$

Of course $\mathbf{M(g)} = \mathbf{Syst^0(g)}$. Recall from Section 4.3 that $M(r, D, a)_i$ is the stratum of $M(r, D, a)$ of sheaves E with $h^0(E) = i$. Note that highest dimensional stratum is $i = r + a = \chi(E)$; define a matrix $\mathbf{M^0(g)} = (M^0(g)_{ij})_{i,j \geq 0}$ of the virtual Hodge polynomials of these generic strata:

$$M^0(g)_{ij} = \begin{cases} \chi_{t,\bar{t}}(M(\frac{i-j}{2}, g, \frac{i+j}{2})_i & i - j \equiv 0 \mod 2 \\ 0 & i - j \equiv 1 \mod 2 \end{cases}$$

6.2 Encoding the Geometry

For any locally closed stratification of a scheme X, the virtual Hodge polynomial of X is the sum of the virtual Hodge polynomials of the strata. In particular,

$$\chi_{t,\bar{t}}(M(r, D, a)) = \sum_{i=0}^{\infty} \chi_{t,\bar{t}}(M(r, D, a)_i) \tag{6.1}$$

Of course the terms are zero until $i = \min(0, r + a)$. Similarly

$$\chi_{t,\bar{t}}(\mathrm{Syst}^n(r, D, a)) = \sum_{i=0}^{\infty} \chi_{t,\bar{t}}(\mathrm{Syst}^n(r, D, a)_i)$$

Recall from Section 4.5 that there is a diagram for $0 \leq r, i \leq n$,

$$\begin{array}{ccc} & \mathrm{Syst}^n(r, D, a)_i & \\ & {\scriptstyle p}\swarrow \quad \searrow{\scriptstyle q} & \\ M(r, D, a)_i & & M(r-n, D, a-n)_{i-n} \end{array}$$

which can be rewritten for $i, r, n \geq 0$ as

$$\begin{array}{ccc} & \mathrm{Syst}^n(r+n, D, a+n)_{i+n} & \\ & {\scriptstyle p}\swarrow \quad \searrow{\scriptstyle q} & \\ M(r+n, D, a+n)_{i+n} & & M(r, D, a)_i \end{array}$$

Recall that the fiber of p above $M(r+n, D, a+n)_{i+n}$ is $\mathrm{Gr}(n, i+n)$ and the fiber of q over $M(r, D, a)_i$ is $\mathrm{Gr}(n, i-r-a)$ ($i \geq r + a$ since $h^0(E) \geq \chi(E)$ for any stable E as $h^2(E) = 0$). Taking $n = i - r - a$, we have

$$\begin{array}{ccc} & \mathrm{Syst}^{i-r-a}(i-a, D, i-r)_{2i-r-a} & \\ & {\scriptstyle p}\swarrow \quad \searrow{\scriptstyle q} & \\ M(i-a, D, i-r)_{2i-r-a} & & M(r, D, a)_i \end{array}$$

where q is an isomorphism and p is an étale-locally fibration with fiber $\mathrm{Gr}(i - r - a, 2i - r - a)$.

For any Zariski-locally trivial fibration $Y \to S$ with fiber F—*i.e.* Zariski-locally trivially on S, $Y \to S$ is isomorphic to the projection $F \times S \to S$—the Hodge polynomials simply multiply

$$\chi_{t,\bar{t}}(Y) = \chi_{t,\bar{t}}(F)\chi_{t,\bar{t}}(S)$$

The same is not in general true for étale-locally trivial fibrations, but it is in this case:

Lemma 6.2.1. *Let Y, S be smooth k-schemes, S simply connected, and $\pi : Y \to S$ a projective morphism that is an étale-locally trivial fibration with fiber* $\mathrm{Gr}(k,n)$*. Then*

$$\chi_{t,\bar{t}}(Y) = \begin{bmatrix} n \\ k \end{bmatrix} \chi_{t,\bar{t}}(S)$$

Proof. Let $\underline{\mathbb{Q}_\ell}$ be the ℓ-adic constant sheaf in the étale topology. The Leray spectral sequence $H^p_{\acute{e}t}(S, R^q\pi_*\underline{\mathbb{Q}_\ell}) \Rightarrow H^{p+q}_{\acute{e}t}(Y, \underline{\mathbb{Q}_\ell})$ associated to π degenerates on the $n = 2$ page. S is simply connected, so the local systems $R^q\pi_*\underline{\mathbb{Q}_\ell}$ are trivial. □

Thus,

$$\begin{aligned} \chi_{t,\bar{t}}(M(r,D,a)_i) &= \chi_{t,\bar{t}}(\mathrm{Gr}(i-r-a, 2i-r-a))\chi_{t,\bar{t}}(M(i-a,D,i-r)_{2i-r-a}) \\ &= \begin{bmatrix} 2i-r-a \\ i-r-a \end{bmatrix} \chi_{t,\bar{t}}(M(i-a,D,i-r)_{2i-r-a}) \end{aligned}$$

After replacing $\ell = r$, $a = k+\ell$, and $i = k+2\ell+s$, this becomes

$$\chi_{t,\bar{t}}(M(\ell,D,k+\ell)_{k+2\ell+s}) = \begin{bmatrix} k+2\ell+2s \\ s \end{bmatrix} \chi_{t,\bar{t}}(M(\ell+s,D,k+\ell+s)_{k+2\ell+2s})$$

The Hodge polynomial on the right is $M^0(g)_{k+2\ell+2s,k}$. The strata $M(\ell,D,k+\ell)_{k+2\ell+s}$

are null below $s = 0$, so

$$\begin{aligned} M(g)_{k+2\ell,k} &= \chi_{t,\bar{t}}(M(\ell, D, k+\ell)) \\ &= \sum_{s=0}^{\infty} \begin{bmatrix} k+2\ell+2s \\ s \end{bmatrix} M^0(g)_{k+2\ell+2s,k} \\ &= \sum_{s=0}^{\infty} A^0_{k+2\ell,k+2\ell+2s} M^0(g)_{k+2\ell+2s,k} \end{aligned}$$

and thus

$$\mathbf{M(g)} = \mathbf{A(0)M^0(g)}$$

Moreover, since

$$\chi_{t,\bar{t}}(\mathrm{Syst}^n(r, D, a)_i) = \chi_{t,\bar{t}}(\mathrm{Gr}(n, i))\chi_{t,\bar{t}}(M(r, D, a)_i)$$

We have

$$\chi_{t,\bar{t}}(\mathrm{Syst}^n(\ell, D, k+\ell)_{k+2\ell+s}) = \begin{bmatrix} k+2\ell+s \\ n \end{bmatrix} \chi_{t,\bar{t}}(M(\ell, D, k+\ell)_{k+2\ell+s})$$

so that

$$\begin{aligned} \mathrm{Syst}^n(g)_{k+2\ell,k} &= \chi_{t,\bar{t}}(\mathrm{Syst}^n(\ell, D, k+\ell)) \\ &= \sum_{s=0}^{\infty} \begin{bmatrix} k+2\ell+s \\ n \end{bmatrix} \begin{bmatrix} k+2\ell+2s \\ s \end{bmatrix} M^0(g)_{k+2\ell+2s,k} \\ &= \sum_{s=0}^{\infty} A^n_{k+2\ell,k+2\ell+2s} M^0(g)_{k+2\ell+2s,k} \end{aligned}$$

and

$$\mathbf{Syst^n(g)} = \mathbf{A(n)M^0(g)}$$

Thus,

Proposition 6.2.2.

$$\mathbf{Syst^n(g) = A(n)A(0)^{-1}M(g) = P(n)M(g)}$$

6.3 Explicit Computations

By (4.2.10), $M(r, D, a)$ is deformation equivalent to the Hilbert scheme of points $X^{[g-ra]}$, so

$$\chi_{t,\bar{t}}(M(r, D, a)) = \chi_{t,\bar{t}}(X^{[g-ra]})$$

The generating function for the Hodge polynomials of the $X^{[n]}$ is, by Göttsche's formula [Göt90]

$$\begin{aligned}\sum_{n\geq 0}\chi_{t,\bar{t}}(X^{[n]})q^n &= \prod_{n\geq 1}\prod_{i,j=0}^{2}(1-(-1)^{i+j}t^{i-1}\bar{t}^{j-1}(uq)^n)^{-(-1)^{i+j}h^{i,j}(X)} \\ &= \prod_{n\geq 1}\frac{1}{(1-u^{-1}(uq)^n)(1-t\bar{t}^{-1}(uq)^n)(1-(uq)^n)^{20}(1-\bar{t}t^{-1}(uq)^n)(1-u(uq)^n)}\end{aligned}$$

Or more simply put,

$$\sum_{n\geq 0}\chi_{t,\bar{t}}(X^{[n]})u^{-n}q^n = \prod_{n\geq 1}\frac{1}{(1-u^{-1}q^n)(1-t^2u^{-1}q^n)(1-q^n)^{20}(1-t^{-2}uq^n)(1-uq^n)} \tag{6.2}$$

Denote by $c(n) = \chi_{t,\bar{t}}(X^{[n]})$. We are interested in the generating function

$$\begin{aligned} F_n^r(y,q) =& \sum_{g\geq 0}\sum_{k\in\mathbb{Z}} \chi_{t,\bar{t}}(\mathrm{Syst}^n(r,D_g,k+r))u^{-g}y^kq^g \\ =& \sum_{g\geq 0}\sum_{k\geq 0} \chi_{t,\bar{t}}(\mathrm{Syst}^n(r,D_g,k+r))u^{-g}y^kq^g \\ &+ \sum_{g\geq 0}\sum_{k<0} \chi_{t,\bar{t}}(\mathrm{Syst}^n(r,D_g,k+r))u^{-g}y^kq^g \end{aligned} \tag{6.3}$$

For $r \leq n$, we know by (4.5.4) that

$$\mathrm{Syst}^n(r,D,r-k) \cong \mathrm{Syst}^n(n-r,D,n-r+k)$$

and therefore we can write (6.3) as

$$F_n^r(y,q) = \sum_{g\geq 0}\sum_{k\geq 0} \mathrm{Syst}^n(g)_{k+2r,k}u^{-g}y^kq^g + \sum_{g\geq 0}\sum_{k>0} \mathrm{Syst}^n(g)_{k+2(n-r),k}u^{-g}y^{-k}q^g$$

We have

$$\begin{aligned} \mathrm{Syst}^n(g)_{k+2r,k} &= \sum_{\ell\geq r} P^n_{k+2r,k+2\ell}M(g)_{k+2\ell,k} \\ &= \sum_{\ell\geq r} P^n_{k+2r,k+2\ell}c\left(g-\ell^2-lk\right) \\ \mathrm{Syst}^n(g)_{k+2(n-r),k} &= \sum_{\ell\geq r+n} P^n_{k-2r+2n,k+2\ell}M(g)_{k+2\ell,k} \\ &= \sum_{\ell\geq r+n} P^n_{k-2r+2n,k+2\ell}c\left(g-\ell^2-lk\right) \end{aligned}$$

Therefore

$$\begin{aligned}F_n^r(q,y) &= \sum_{g\geq 0}\sum_{k\in\mathbb{Z}} u^{-g}q^g y^k \operatorname{Syst}^n(g)_{k+2r,k} \\ &= \sum_{g\geq 0}\sum_{k\geq 0} u^{-g}q^g y^k \sum_{\ell\geq r} P^n_{k+2r,k+2\ell}c(g-\ell^2-lk) \\ &\quad + \sum_{g\geq 0}\sum_{k\geq 1} u^{-g}q^g y^{-k} \sum_{\ell\geq n-r} P^n_{k-2r+2n,k+2\ell}c(g-\ell^2-lk)\end{aligned}$$

and thus

$$F_n^r(q,y) = S\sum_{k\geq 0}\sum_{\ell\geq r} y^k u^{-\ell^2-lk} q^{lk+\ell^2} P^n_{k+2r,k+2\ell} \tag{6.4}$$

$$+ S\sum_{k\geq 1}\sum_{\ell\geq n-r} y^{-k} u^{-\ell^2-lk} q^{lk+\ell^2} P^n_{k-2r+2n,k+2\ell} \tag{6.5}$$

where

$$S = \sum_{g\geq 0} c(g)u^{-g}q^g$$

is the generating function of the Hodge polynomials of the Hilbert schemes of points. We also know by (5.5.1) that

$$\begin{aligned}P^n_{k+2r,k+2\ell} &= u^{r(n-r)}u^{\ell^2+lk-nl-kr}\frac{[k+2\ell]}{[n]}\begin{bmatrix}n+\ell-r-1\\ n-1\end{bmatrix}\begin{bmatrix}k+\ell+r-1\\ n-1\end{bmatrix} \\ P^n_{k-2r+2n,k+2\ell} &= u^{r(n-r)}u^{\ell^2+lk-nl-k(n-r)}\frac{[k+2\ell]}{[n]}\begin{bmatrix}\ell+r-1\\ n-1\end{bmatrix}\begin{bmatrix}k+\ell-r+n-1\\ n-1\end{bmatrix}\end{aligned}$$

Note that the sums in (6.4), (6.5) make sense for all $\ell \geq 0$ since the terms are zero whenever $\ell < r$ in the first and $\ell < n-r$ in the second sum.

Write $p = \ell + k$ to get

$$\sum_{k\geq 0}\sum_{\ell\geq 0} y^k u^{-\ell^2-lk} q^{lk+\ell^2} P^n_{k+2r,k+2\ell}$$
$$= \frac{u^{r(n-r)}}{[n]} \sum_{\ell\geq 0}\sum_{p\geq \ell} u^{-nl-(p-\ell)r}[p+\ell] \begin{bmatrix} n+\ell-r-1 \\ n-1 \end{bmatrix} \begin{bmatrix} p+r-1 \\ n-1 \end{bmatrix} y^{p-\ell} q^{pl}$$

and

$$\sum_{k>0}\sum_{\ell\geq 0} y^{-k} u^{-\ell^2-lk} q^{lk+\ell^2} P^n_{k-2r+2n,k+2\ell}$$
$$= \frac{u^{r(n-r)}}{[n]} \sum_{p>\ell}\sum_{\ell\geq 0} u^{-np+(p-\ell)r}[p+\ell] \begin{bmatrix} \ell+r-1 \\ n-1 \end{bmatrix} \begin{bmatrix} n-r+p-1 \\ n-1 \end{bmatrix} y^{\ell-p} q^{pl}$$
$$\overset{\ell\leftrightarrow p}{=} \frac{u^{r(n-r)}}{[n]} \sum_{p\geq 0}\sum_{\ell>p} u^{-nl-(p-\ell)r}[p+\ell] \begin{bmatrix} n+\ell-r-1 \\ n-1 \end{bmatrix} \begin{bmatrix} p+r-1 \\ n-1 \end{bmatrix} y^{p-\ell} q^{pl}$$

Where the second line is obtained by setting $k = p - \ell$, and the third by switching p and ℓ. The result is

Theorem 6.3.1. *For $r \leq n$*

$$S^{-1}F^r_n(q,y) \quad = \quad \frac{u^{r(n-r)}}{[n]} \sum_{\ell\geq 0}\sum_{p\geq 0} u^{-nl-(p-\ell)r}[p+\ell] \begin{bmatrix} n+\ell-r-1 \\ n-1 \end{bmatrix} \begin{bmatrix} p+r-1 \\ n-1 \end{bmatrix} y^{p-\ell} q^{pl}$$

Note that the only dependence on $t, \bar{t}$ that doesn't factor through $u = t\bar{t}$ is from the term S. In particular for $r = 0, n = 1$

$$\begin{aligned} S^{-1}F_1^0(q,y) &= \sum_{\ell \geq 0} \sum_{p \geq 1} u^{-\ell}[p+\ell] y^{p-\ell} q^{p\ell} \\ &= \frac{1}{u-1} \sum_{\ell \geq 0} \sum_{p \geq 1} (u^p - u^{-\ell}) y^{p-\ell} q^{p\ell} \\ &= \frac{1}{u-1} \Psi(u,y) \end{aligned}$$

and

Corollary 6.3.2.

$$S^{-1}F_1^0(q,y) = \frac{-1}{(1-y)(1-u^{-1}y^{-1})} \prod_{n \geq 1} \frac{(1-q^n)^2(1-uq^n)(1-u^{-1}q^n)}{(1-yq^n)(1-y^{-1}q^n)(1-uyq^n)(1-u^{-1}y^{-1}q^n)}$$

Note directly from the formula in (6.3.1) that the duality (4.5.4) manifests itself in a kind of rank-level duality for the generating function $F_n^r(q,y)$:

Corollary 6.3.3.

$$F_n^r(q,y) = F_n^{n-r}(q,y^{-1})$$

6.4 Relation to $r = 0, n = 1$

The higher generating functions are actually determined by the $r = 0, n = 1$ function. Define Laurent polynomials $B_n^r(i,j)$ in u for $n \geq 1$, $1 \leq i \leq n$ and $0 \leq j \leq n-i$ by $B_n^r(n,0) = 1$ and

$$B_{n+1}^r(i,j) = B_n^r(i-1,j) + B_n^r(i+1,j-1) - u^{r-n} B_n^r(i,j-1) - u^{n-r} B_n^r(i,j)$$

Lemma 6.4.1.

$$u^{-n\ell-(p-\ell)r}[p+\ell]\begin{bmatrix} n+\ell-r-1 \\ n-1 \end{bmatrix}\begin{bmatrix} p+r-1 \\ n-1 \end{bmatrix} = \frac{(u-1)^{1-2n}}{[n-1]!^2}\sum_{i=1}^{n}\sum_{j=0}^{n-i} B_n^r(i,j)(u^{ip}-u^{-i\ell})u^{j(p-\ell)}$$

Proof. Clearly the claim is true for $n = 1$. Note that

$$\begin{aligned}\frac{u^{-\ell}[n+\ell-r][p+r-n]}{[n]^2} &= \frac{(u-1)^2}{[n]^2}(u^{n-r}-u^{-\ell})(u^{p+r-n-\ell}-u^{\ell}) \\ &= \frac{(u-1)^2}{[n]^2}(u^p - u^{p+r-n-\ell} - u^{n-r} + u^{-\ell})\end{aligned}$$

Thus by induction

$$\begin{aligned}&u^{-(n+1)\ell-(p-\ell)r}[p+\ell]\begin{bmatrix} n+\ell-r-1 \\ n \end{bmatrix}\begin{bmatrix} p+r-1 \\ n \end{bmatrix} \qquad (6.6)\\ &= \frac{u^{-\ell}[n+\ell-r][p+r-n]}{[n]^2}\left(u^{-n\ell-(p-\ell)r}[p+\ell]\begin{bmatrix} n+\ell-r-1 \\ n-1 \end{bmatrix}\begin{bmatrix} p+r-1 \\ n-1 \end{bmatrix}\right) \\ &= \frac{(u-1)^2}{[n]^2}(u^p - u^{p+r-n-\ell} - u^{n-r} + u^{-\ell})\left(\frac{(u-1)^{2-2n}}{[n-1]!^2}\sum_{i,j} B_n^r(i,j)(u^{ip}-u^{-i\ell})u^{j(p-\ell)}\right)\end{aligned}$$

Clearly the two fractions match up to give the coefficient we want. Note that

$$\begin{aligned}(u^p+u^{-\ell})(u^{ip}-u^{-i\ell})u^{j(p-\ell)} &= (u^{(i+1)p}-u^{p-i\ell})u^{j(p-\ell)} + (u^{ip-\ell}-u^{-(i+1)\ell})u^{j(p-\ell)} \\ &= (u^{(i+1)p}-u^{-(i+1)\ell})u^{j(p-\ell)} + (u^{(i-1)p}-u^{-(i-1)\ell})u^{(j+1)(p-\ell)}\end{aligned}$$

and

$$-(u^{p+r-n-\ell}+u^{n-r})(u^{ip}-u^{-i\ell})u^{j(p-\ell)} = -u^{r-n}(u^{ip}-u^{-i\ell})u^{(j+1)(p-\ell)} - u^{n-r}(u^{ip}-u^{-i\ell})u^{j(p-\ell)}$$

So that in (6.6) the coefficient of $(u^{ip} - u^{-i\ell})u^{p-\ell}$ is

$$B_n^r(i-1,j) + B_n^r(i+1,j-1) - u^{r-n}B_n^r(i,j-1) - u^{n-r}B_n^r(i,j)$$

which by definition is $B_{n+1}^r(i,j)$.

□

By (6.3.1),

$$\begin{aligned}
[n]u^{r(r-n)}S^{-1}F_n^r(q,y) &= \sum_{\ell\geq 0}\sum_{p\geq 0} u^{-nl-(p-\ell)r}[p+\ell]\begin{bmatrix} n+\ell-r-1 \\ n-1 \end{bmatrix}\begin{bmatrix} p+r-1 \\ n-1 \end{bmatrix} y^{p-\ell}q^{pl} \\
&= \frac{(u-1)^{2-2n}}{[n-1]!^2}\sum_{i=1}^{n}\sum_{j=0}^{n-i} B_n^r(i,j)\sum_{p,\ell\geq 0}(u^{ip}-u^{-i\ell})u^{j(p-\ell)}y^{p-\ell}q^{p\ell} \\
&= \frac{(u-1)^{2-2n}}{[n-1]!^2}\sum_{i=1}^{n}\sum_{j=0}^{n-i} B_n^r(i,j)\Psi(u^i,u^jy)
\end{aligned}$$

So finally

Theorem 6.4.2.

$$S^{-1}F_n^r(q,y) = \frac{u^{r(n-r)}(u-1)^{2-2n}}{[n][n-1]!^2}\sum_{i=1}^{n}\sum_{j=0}^{n-i} B_n^r(i,j)\Psi(u^i,u^jy)$$

For example for $n=2$ the only nonzero $B_2^r(i,j)$ are

$$B_2^r(2,0) = 1 \quad B_2^r(1,0) = -u^{1-r} \quad B_2^r(1,1) = -u^{r-1}$$

and therefore

$$\frac{u^{r(r-2)}(u-1)^2[2]}{S}F_2^r(q,y) = \Psi(u^2,y) - u^{1-r}\Psi(u,y) - u^{r-1}\Psi(u,uy)$$

6.5 Euler Characteristics and Modularity

Of particular interest is the generating function $F_n^r(q,y)|_{u=1}$ of the Euler characteristics of the moduli spaces $\mathrm{Syst}^n(r,D,a)$. By definition,

$$F_n^r(q,y)|_{u=1} = \sum_{g\geq 0}\sum_{k\in\mathbb{Z}} \chi_{t,\bar{t}}(\mathrm{Syst}^n(r,D_g,k))y^k q^g$$

The generating function $S|_{u=1}$ of the Euler characteristics of the Hilbert scheme of points is well known. From (6.2):

$$S|_{u=1} = \sum_{n\geq 0}\chi(X^{[n]})q^n = \prod_{n\geq 1}\frac{1}{(1-q^n)^{24}} = \frac{1}{q\eta(q)^{24}}$$

where $\eta(q)$ is the q-expansion of the Dedekind η function. Define

$$G_n^r(q,y) := q\eta(q)^{24}F_n^r(q,y)|_{u=1}$$

From (6.3.1),

Theorem 6.5.1.

$$G_n^r(q,y) = \frac{1}{n}\sum_{\ell\geq 0}\sum_{p\geq 1}(p+\ell)\binom{n+\ell-r-1}{n-1}\binom{p+r-1}{n-1}y^{p-\ell}q^{pl}$$

Note that the coefficient in (6.3.2) can be rewritten at $u=1$ as

$$\frac{-1}{(1-y)(1-y^{-1})} = \left(\sqrt{y}-\frac{1}{\sqrt{y}}\right)^{-2}$$

Thus, for $r=0$, $n=1$ we recover the Kawai-Yoshioka formula [KY00]

Corollary 6.5.2.

$$G_1^0(q,y) = \left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)^{-2} \prod_{n\geq 1} \frac{(1-q^n)^4}{(1-yq^n)^2(1-y^{-1}q^n)^2}$$

We can in fact express all of the higher generating functions $G_n^r(q,y)$ in terms of the Kawai-Yoshioka function $G_1^0(q,y)$ via differential operators. Define for $r \leq n$

$$D_n^r = \frac{1}{n(n-1)}\left(q\frac{d}{dq} + (n-r-1)y\frac{d}{dy} - (n-r-1)^2\right)$$

Note that

$$\begin{aligned}\binom{n+\ell-r-1}{n-1}\binom{p+r-1}{n-1} &= \\ &= \frac{(\ell+(n-r-1))\cdots(\ell-r+1)}{(n-1)!}\frac{(p+r-1)\cdots(p-(n-r-1))}{(n-1)!} \\ &= \frac{(\ell+(n-r-1))(p-(n-r-1))}{(n-1)^2}\binom{n+\ell-r-2}{n-2}\binom{p+r-1}{n-2}\end{aligned}$$

and since $(\ell+(n-r-1))(p-(n-r-1)) = p\ell + (p-\ell)(n-r-1) - (n-r-1)^2$, we have

Theorem 6.5.3. *For $r \leq n$, $n \geq 2$*

$$G_n^r(q,y) = D_n^r G_{n-1}^r(q,y)$$

As noted by [KY00], the function $G_1^0(q,y)$ is itself modular. Recall (*cf.* [Fol09]) that the Igusa cusp form χ_{10} is the unique (up to normalization) weight 10 Siegel modular form on the Siegel space

$$\mathbb{H}_2 = \{\Omega = \left(\begin{smallmatrix}\tau' & \nu \\ \nu & \tau\end{smallmatrix}\right) \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) | \mathrm{Im}\,\Omega > 0\}$$

The nth coefficient of the Fourier expansion in the variable $q' = e^{2\pi i \tau'}$

$$\chi_{10} = \sum_{n \geq 0} (q')^n \chi_{10,n}(\tau, \nu)$$

is a Fourier-Jacobi form of weight 10 and index n. If we write $\chi_{10,n}(q, y)$ for the Fourier expansion of $\chi_{10,n}(\tau, \nu)$ as a formal power series in $q = e^{2\pi i \tau}$ and $y = e^{2\pi i \nu}$, then

Theorem 6.5.4. *[KY00]*

$$\chi_{10,1}(q, y) = G_1^0(q, y)$$

Using (6.3.3) and (6.5.3) we can express the higher generating functions in terms of the derivatives of a Fourier-Jacobi modular form; this can be viewed as an analog of the fact that the Gromov-Witten potentials of the K3 surface are all quasimodular forms.

Chapter 7

Concluding Remarks

7.1 Insertions

Though the equivalence between the full Gromov-Witten and Donaldson-Thomas theories with insertions is conjectured to be true for all threefolds, it is only known in two special cases: (i) Calabi-Yau toric threefolds in full generality [MNOP06a, MNOP06b]; (ii) arbitrary toric threefolds for primary insertions [MOOP]. The Gromov-Witten theory of surfaces already has a well defined notion of insertions, and the full theory is calculated for the K3 surface in [MPT]. One can ask how to construct invariants with insertions for stable pairs on surfaces.

Let X be a K3 surface, D a divisor class of genus g on X, and $\mathcal{C}^{[d]}$ the hilbert scheme of d points on the universal divisor $\mathcal{C} \to \mathbb{P} = |D| \cong \mathbb{P}^g$. As discussed in the introduction, $P_{n,g} = \mathcal{C}^{[n+g-1]} \cong \mathrm{Syst}^1(0, D, n)$ is the analog of the Pandharipande-Thomas moduli space of stable pairs. There is an obvious map $\rho : P_{n,g} \to |D| \cong \mathbb{P}^g$. Letting H be the hyperplane class on $\mathbb{P}^g$, invariants with insertions can be defined by integrating the fundamental class of $\mathcal{C}^{[d]}$ against the Chern classes of the cotangent bundle $\Omega_{P_n^g}$ and powers of $\rho^* H$:

$$C_{n,g}^k = \int_{P_{n,g}} c_{n+2g-1-k}(\Omega_{P_n^g}) \cup \rho^*(H^k) \tag{7.1}$$

The generating function of the $C_{n,g}^k$ is computed in [MPT] and related to the Gromov-Witten theory of X. The invariant (7.1) is naively the Euler characteristic of the moduli space of divisors in the class D meeting k fixed general points on X.

In the higher rank case, there is no longer a map to $\mathbb{P}^g$, but using $\rho : \mathrm{Syst}^n(r, D, a) \to M(r, D, a)$, one can define invariants with insertions

$$\int_{\mathrm{Syst}^n(r,D,a)} c_{N-k}(\Omega_{\mathrm{Syst}^n(r,D,a)})\rho^* x \tag{7.2}$$

where $N = \dim \mathrm{Syst}^n(r, D, a)$ and x is a cohomology class of degree k on $M(r, D, a)$. Since $M(r, D, a)$ is deformation equivalent to $X^{[g-ra]}$, the cohomology of $M(r, D, a)$ is well-understood. The precise relationship between the Gromov-Witten theory of X and the invariants (7.2) has yet to be understood, but may shed some light on the relationship between higher rank sheaf-theoretic virtual counts and Gromov-Witten theory of threefolds.

7.2 Abelian Surfaces

The moduli of sheaves on abelian surfaces is quite similar to that of K3 surfaces. Let A be an abelian surface over k.

- The stable locus M^s in the moduli space M of semistable sheaves is smooth by (4.2.1).

- For a divisor class D of minimal degree on A, it will still be true that μ-semistability implies μ-stability for sheaves $\mathcal{E}$ with $v(\mathcal{E}) = (r, D, a)$, and therefore $M(v) = M^s(v)$ is smooth and projective.

- Serre duality gives a canonical holomorphic symplectic form on $M(v)$, making $M(v)$ an irreducible symplectic variety.

- Let $A^{[n]}$ be the hilbert scheme of n points on A; it will *not* be true that each $M(v)$ is deformation equivalent to some $A^{[n]}$. If, however, we define the *generalized Kummer variety*[1] $K_{n-1}(A)$, the fiber of the addition map $A^{[n]} \to A$, then each $M(v)$ is deformation equivalent to some $K_{n-1}(A)$.

It is expected that the computation in Chapter 6 can be adapted to the abelian surface case, and that this will similarly agree with the reduced Gromov-Witten theory of A. This will be pursued in an upcoming paper with Andrei Jorza.

7.3 Stability Conditions

One can also vary the notion of stability used to pick out the moduli spaces $M(r, D, a)$, and ask how the computation of Chapter 6 changes. In [Bri07], Bridgeland defines a notion of stability condition on the bounded derived category $D^b(X)$ of coherent sheaves on a smooth variety X. For X a threefold, it is hoped that one can view the Donaldson-Thomas and Pandharipande-Thomas moduli spaces as moduli spaces of stable objects in $D^b(X)$ with respect to two different stability conditions, and that the equivalence between the two manifests itself as a wall-crossing formula. Much progress has been made toward understanding this insight, but stability conditions and the moduli of stable objects with respect to them have proved difficult to construct. On the other hand, numerical invariants of weaker stability conditions on $D^b(X)$ have been constructed directly by Joyce [Joy06, Joy07a, Joy07b, Joy08] and Kontesevich and Soibelman [KS], and the equivalence has been proven from this viewpoint in many contexts by many authors, for example Joyce, Kontsevich and Soibelman in the aforementioned references, Toda in [Tod], and [Bri].

[1] So called because $K_1(A)$ is the Kummer surface associated to A.

In contrast to threefolds, many Bridgeland stability conditions have been constructed on K3 surfaces in [Bri08], and the moduli spaces of stable objects have been constructed for some of them [ABL]. Most of the properties of stable sheaves used in Chapter 4 to construct the diagram that facilitated the computation in Chapter 6 carry over to Bridgeland stable objects, and it would be interesting to see whether there is an explicit wall-crossing formula relating the resulting invariants.

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