HIGHER RANK STABLE PAIRS ON K3 SURFACES

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Abstract. We define and compute higher analogs of Pandharipande-Thomas stable pair invariants for K3 surfaces. Curve-counting on K3 surfaces is already well developed; their reduced Gromov-Witten theory has been computed in primitive classes by Maulik, Pandharipande, and Thomas. The partition functions are quasi-modular forms, and there is a MNOP-style equivalence via a change of variable with generating functions of Euler characteristics of moduli spaces of rank $r = 0$ stable pairs with $n = 1$ sections. The Euler characteristics of the stable pair moduli spaces for higher rank $r \geq 0$ and section rank $n \geq 1$ are naturally interpreted as a higher stable pair invariant. We fully compute the Hodge polynomials and Euler characteristics of these moduli spaces, prove that the resulting partition functions are modular forms, and explore the relationship of the higher invariants to Gromov-Witten theory.

1. Introduction

1.1. **Curve Counting Correspondences for Threefolds.** There are to date three methods of virtually counting pointed curves on a smooth algebraic threefold X (over $\mathbb{C}$), each by integrating against the virtual fundamental class of an appropriately defined moduli space of curves M. Given a curve class $\beta \in H_2(X, \mathbb{Z})$:

- Gromov-Witten theory uses the moduli stack $M = \overline{M}_{g,n}(X, \beta)$ [KM94] of n-pointed genus g stable maps with image in the class β. The stack carries a perfect obstruction theory [BF97], and the resulting virtual class has dimension

$$\dim[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}} = \int_\beta \mathrm{c}_1(X) + (3 - \dim X)(g-1) + n = \int_\beta \mathrm{c}_1(X) + n \qquad (1)$$

- Donaldson-Thomas theory [Tho00] uses the Hilbert scheme $M = I_k(X, \beta)$ of subschemes $Z \subset X$ with $[Z] = \beta$ and $\chi(\mathcal{O}_Z) = k$. The virtual dimension is

$$\dim[I_k(X,\beta)]^{\mathrm{vir}} = \int_\beta \mathrm{c}_1(X)$$

- Pandharipande-Thomas theory [PT09b, PT09a, PT10] uses the moduli scheme $M = P_k(X, \beta)$ of stable pairs $\mathcal{O} \to \mathcal{F}$ in the sense of Le Potier [LP93, LP95],

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where $\mathcal{F}$ is a pure sheaf supported on a curve in the class β and $\chi(F) = k$. The virtual dimension is also

$$\dim[P_k(X,\beta)]^{\mathrm{vir}} = \int_\beta \mathrm{c}_1(X)$$

In each case, there are naturally defined cohomology classes on M, and the invariants produced by integrating them against the virtual fundamental class provide a virtual count of curves satisfying the associated incidence conditions. For example, in Gromov-Witten theory there is a universal curve $\pi : \mathcal{C} \to \overline{M}_{g,n}(X,\beta)$ and a universal stable map $\mu : \mathcal{C} \to X$. There are n sections $\sigma_i : \overline{M}_{g,n}(X,\beta) \to \mathcal{C}$ corresponding to the n marked points; the ith evaluation map is the composition

$$\mathrm{ev}_i = \mu\sigma_i : \overline{M}_{g,n}(X,\beta) \to X$$

which on the level of $\mathbb{C}$-points maps a stable curve $f : C \to X$ with ith marked point $p_i \in C$ to $f(p_i)$. $\mathcal{C}$ is a flat curve over $\overline{M}_{g,n}(X,\beta)$ with only nodal singularities, so the relative dualizing sheaf $\omega = \omega_\pi$ is a line bundle, and we define $\psi_i = \mathrm{c}_1(\sigma_i^*\omega)$ and $\lambda_i = c_i(\pi_*\omega)$.

For X a Calabi-Yau variety ($\mathrm{c}_1(X) = 0$), the dimension of all three moduli spaces is 0 for any g, k (and $n = 0$ in Gromov-Witten theory), and integrating the unit class yields a number. For Gromov-Witten theory, we have

$$N^{\mathrm{GW}}_{g,\beta} = \int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} 1$$

and similarly we define $N^{\mathrm{AB}}_{k,\beta}$, for $\mathrm{AB} = \mathrm{DT}, \mathrm{PT}$. These numbers are organized into generating functions, though each with a slight subtlety. Pandharipande-Thomas theory is the most straight-forward; the reduced partition function is simply

$$Z'_{\mathrm{PT}}(X;q,v) = \sum_{\substack{\beta\in H_2(X,\mathbb{Z})\\ \beta\neq 0}} \sum_{k\in\mathbb{Z}} N^{\mathrm{PT}}_{k,\beta} q^k v^\beta$$

Similarly, we form the Donaldson-Thomas partition function

$$Z_{\mathrm{DT}}(X;q,v) = \sum_{\beta\in H_2(X,\mathbb{Z})} \sum_{k\in\mathbb{Z}} N^{\mathrm{DT}}_{k,\beta} q^k v^\beta$$

Moduli points in Donaldson-Thomas theory are 1-dimensional subschemes that must have a component supported on a curve in the class β, but may include points off of that curve. To correct for this, the reduced Donaldson-Thomas partition function is obtained by dividing out the degree zero contribution,

$$Z'_{\mathrm{DT}}(X;q,v) = \frac{Z_{\mathrm{DT}}(X;q,v)}{Z_{\mathrm{DT}}(X;q)_0}$$

where

$$Z_{\mathrm{DT}}(X;q)_0 = \sum_{n\in\mathbb{Z}} N^{\mathrm{DT}}_{k,0} q^k$$

It has been computed; let

$$M(q) = \prod_{k\geq 0} \frac{1}{(1-q^k)^k}$$

be the McMahon function, the generating function of 3-dimensional partitions. Then we have

Theorem 1.1. *[Li06, BF08, LP]. For X a threefold*

$$Z_{\mathrm{DT}}(X;q)_0 = M(-q)^{\chi(X)}$$

where $\chi(X)$ is the topological Euler characteristic.

The correspondence between the two sheaf-counting theories is easy:

Theorem 1.2.

$$Z'_{\mathrm{DT}}(X;q,v) = Z'_{\mathrm{PT}}(X;q,v)$$

Theorem (1.2) was treated in the toric case by [PT09a]; it was observed in [PT09b] that the equality can be viewed as a wall-crossing formula for invariants of stability conditions on $D^b(X)$. The general case of the theorem has been treated by many authors, see [Tod, ST, Bri, KS, JS].

Finally, the reduced Gromov-Witten potential

$$F'_{\mathrm{GW}}(X;u,v) = \sum_{\substack{\beta\in H_2(X,\mathbb{Z})\\ \beta\neq 0}} \sum_{g\geq 0} N^{\mathrm{GW}}_{g,\beta} u^{2g-2} v^{\beta}$$

generates Gromov-Witten invariants, all of which count irreducible curves. The reduced Gromov-Witten partition function

$$Z'_{\mathrm{GW}}(X;u,v) = \exp F'_{\mathrm{GW}}(X;u,v)$$

generates Gromov-Witten invariants with possibly disconnected domain curves, and we then have

Theorem 1.3. *[MOOP]. For X a toric Calabi-Yau threefold*[1]

$$Z'_{\mathrm{GW}}(X;u,v) = Z'_{\mathrm{DT}}(X;-e^{iu},v)$$

Theorem (1.3) was originally conjectured in [MNOP06a, MNOP06b]. It has also been proven for the generating functions of primary invariants for all toric threefolds [MOOP].

[1]There are no proper toric Calabi-Yau threefolds, but GW and DT invariants can still be defined by equivariant localization for local Calabi-Yau threefolds.

1.2. **The Gromov-Witten Theory of K3 Surfaces.** It is natural to ask whether there is an analogous relationship between Gromov-Witten theory and sheaf-theoretic virtual curve counts on surfaces. In the threefold case, the relationship is most easily described in the Calabi-Yau case, so it is natural to ask the question first for K3 surfaces. The Gromov-Witten theory of K3 surfaces has been studied by Maulik, Pandharipande, and Thomas [MPT].

Let X be a K3 surface, β a curve class on X. The normal Gromov-Witten theory of X vanishes because X can always be deformed so that β is no longer algebraic. This triviality manifests itself as a trivial quotient of the obstruction bundle Obs. Indeed, the obstruction space at a stable map $f : C \to X$ is $H^1(f^*T_X)$, but since $\omega_X \cong \mathcal{O}_X$, the canonical map

$$H^1(f^*T_X) \cong H^1(f^*\Omega^1_X) \to H^1(\omega_C) \cong \mathbb{C}$$

yields a trivial quotient Obs $\to \mathcal{O}$. This forces the virtual class to be 0 since, naively, $[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}$ is Poincaré dual to the Euler class of the obstruction bundle.

After modifying the obstruction theory by taking instead the kernel of Obs $\to \mathcal{O}$ to be the obstruction bundle, we obtain a reduced virtual class $[\overline{M}_{g,n}(X,\beta)]^{\mathrm{red}}$ with virtual dimension one greater than expected:

$$\dim[\overline{M}_{g,n}(X,\beta)]^{\mathrm{red}} = 1 + \int_\beta \mathrm{c}_1(T_X) + (3-\dim X)(g-1) + n = g + n$$

The reduced Gromov-Witten invariants are

$$\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle^{X,\mathrm{red}}_{g,\beta} = \int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^n \psi_i^{k_i} \cup \mathrm{ev}_i^*(\gamma_i) \tag{2}$$

The Gromov-Witten potentials have surprising modularity properties:

Theorem 1.4. *[MPT] Let X be an elliptic K3 surface with section, s the section class and f the fiber class. Each Gromov-Witten potential*

$$F_g^X(\tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)) = \sum_{h\geq 0} \langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle^{X,\mathrm{red}}_{g,s+hf}\, q^{h-1}$$

is the Fourier expansion of a quasimodular form.

One can also define invariants involving the Hodge classes λ_i. Let

$$R_{g,\beta} = \int_{[\overline{M}_g(X,\beta)]^{\mathrm{red}}} (-1)^g \lambda_g$$

and define the partition function

$$Z^{\mathrm{GW}}_\beta(v) = \sum_{g\geq 0} R_{g,\beta} v^{2g-2} \tag{3}$$

$Z^{\mathrm{GW}}_\beta(v)$ only depends on the genus h of β by deformation invariance, and we will also denote (3) by $Z^{\mathrm{GW}}_h(v)$. Note we have replaced the usual variable u with v.

1.3. **Sheaf-Counting on K3 Surfaces.** Let X be a K3 surface and D a divisor class such that every divisor in D is reduced and irreducible of genus g. Following [KY00], let $\mathbb{P} = |D|$ be the complete linear system of D and $X \times \mathbb{P} \supset \mathcal{C}_g \to \mathbb{P}$ the universal divisor. The relative Hilbert scheme $\mathcal{C}_g^{[d]} = \mathrm{Hilb}^d(\mathcal{C}_g/\mathbb{P})$ parametrizing divisors C in the class D and subschemes Z of C of length d is the surface analog of the moduli space $P_k(X, \beta)$ of stable pairs $\mathcal{O} \to \mathcal{F}$ with $\mathrm{c}_1(\mathcal{F}) = D$ and $\chi(\mathcal{F}) = d + 1 - g = k$. $\mathcal{C}_g^{[d]}$ is smooth, so a reasonable replacement for the Pandharipande-Thomas invariant is the (signed) topological Euler characteristic of $\mathcal{C}_g^{[d]}$ [PT10]. Let

$$Z_g^{\mathrm{PT}}(y) = \sum_{d \geq 0} (-1)^{d+g} \chi(\mathcal{C}_g^{[d]}) y^{d+1-g}$$

We then have a stable map-stable pair correspondence analogous to (1.2)

Theorem 1.5. *[MPT].* $Z_g^{\mathrm{GW}}(v) = Z_g^{\mathrm{PT}}(-e^{iv})$

1.4. **Outline.** There is a natural generalization of the stable pair moduli spaces $\mathcal{C}_g^{[d]}$ of a K3 surface X; for a primitive divisor D_g on X with $D_g^2 = 2g-2$, we consider the moduli spaces $\mathrm{Syst}^n(r, D_g, k)$ of stable pairs $\mathcal{O}^n \to \mathcal{E}$ with $\chi(\mathcal{E}) = k$, where now we allow more sections $n \geq 1$ and higher rank $\mathrm{rk}(\mathcal{E}) = r \geq 0$. In the $r = 0, n = 1$ case, we clearly have

$$\mathrm{Syst}^1(0, D_g, k) \cong \mathcal{C}_g^{[k+g-1]}$$

as constructed above. By a theorem of Kawai-Yoshioka [KY00], each $\mathrm{Syst}^n(r, D_g, k)$ is smooth, and we can consider the signed Euler characteristic to be a higher rank sheaf-counting invariant. These invariants are the analog of Sheshmani's higher rank stable pairs on threefolds [She], though in the threefold case only the $r = 0$ case has been developed.

The aim of this paper is to compute the Hodge polynomials $\mathrm{e}(\cdot)$ of these moduli spaces and to explore the stable map-stable pair correspondence for K3 surfaces in higher rank. Assembling these polynomials into generating functions

$$F_n^r(q, y) = \sum_{g \geq 0} \sum_{k \in \mathbb{Z}} \mathrm{e}\left(\mathrm{Syst}^n(r, D_g, k + r)\right) (t\bar{t})^{-g} y^k q^{g-1} \tag{4}$$

the main result is Theorem (3.3):

Main Theorem 1.6. *If* $s(q) = \sum_{n \geq 0} \mathrm{e}(X^{[n]})(t\bar{t})^{-n} q^{n-1}$ *is the generating function of the (symmetrized) Hodge polynomials of the Hilbert schemes* $X^{[n]}$ *of* n *points on* X*, then*

$$\frac{F_n^r(q, y)}{S(q)} = \frac{(t\bar{t})^{r(n-r)}}{[n]} \sum_{\substack{p \geq n-r \\ \ell \geq r}} (t\bar{t})^{-n\ell - (p-\ell)r} [p + \ell] \begin{bmatrix} n + \ell - r - 1 \\ n-1 \end{bmatrix} \begin{bmatrix} p + r - 1 \\ n-1 \end{bmatrix} y^{p-\ell} q^{p\ell}$$

The square binomial coefficients are u-binomial coefficients; see Section 4.1. The same method allows one to compute a formula for the Hodge polynomials of the Brill-Noether strata of all moduli spaces of sheaves with primitive first Chern class on a K3 surface.

Setting $n = 1, r = 0$ we rederive the calculation of [KY00], and in particular the potential $Z_g^{\mathrm{PT}}(y)$:

Corollary 1.7.

$$\sum_{g\geq 0} Z_g^{\mathrm{PT}}(y)q^{g-1} = -\left. F_1^0(q,-y)\right|_{t=\bar{t}=1}$$

This is equation (4) of [MPT].

The outline of the paper is as follows. In Section 2 we recall the moduli theory of stable pairs on a K3 surface X. The key relationship between the relevant moduli spaces is developed in Section 2.3. In Section 3 we compute the generating functions (4) using the geometry from Section 2. In Section 3.4 we express the general invariants in terms of the $r = 0, n = 1$ theory; in Section 3.5, we compute the generating functions of the Euler characteristics and prove that the v-coefficients, after setting $y = -e^{iv}$, are modular forms. The less enlightening computations used in the course of Section 3 are collected in Section 4.

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2. The moduli theory of sheaves and stable pairs on K3 surfaces

Throughout this section, let X be an algebraic K3 surface over $\mathbb{C}$. The Mukai lattice of X is the total cohomology ring $H^*(X,\mathbb{Z})$ together with the pairing

$$(v,w) = -\int_X v^\vee w = \int_X (v_1w_1 - v_0w_2 - v_2w_0)$$

where for $v = v_0 + v_1 + v_2 \in H^*(X,\mathbb{Z})$, $v_i \in H^{2i}(X,\mathbb{Z})$ are the homogeneous components, and similarly for w. We will denote by $\omega \in H^4(X,\mathbb{Z})$ the Poincaré dual to the point class. Using the canonical isomorphisms $H^0(X,\mathbb{Z}) \cong \mathbb{Z}$ and $H^4(X,\mathbb{Z}) \cong \mathbb{Z}$, we will write $v = (r, D, a)$ for integers r, a when $v_0 = r$, $v_1 = D$, $v_2 = a\omega$. Note that

$$\mathrm{Td}(X) = 1 + 2\omega$$

Given a coherent sheaf $\mathcal{E}$ on X, the Mukai vector of $\mathcal{E}$ is

$$\begin{aligned} v(\mathcal{E}) &= \mathrm{ch}\,\mathcal{E}\sqrt{\mathrm{Td}(X)} \\ &= \mathrm{rk}(\mathcal{E}) + \mathrm{c}_1(\mathcal{E}) + \left(\mathrm{rk}(\mathcal{E})\omega + \frac{\mathrm{c}_1(\mathcal{E})^2}{2} - \mathrm{c}_2(\mathcal{E})\right) \\ &= (\mathrm{rk}(\mathcal{E}), \mathrm{c}_1(\mathcal{E}), \chi(\mathcal{E}) - \mathrm{rk}(\mathcal{E})) \end{aligned}$$

by Gronthendieck-Riemann-Roch. The Mukai pairing is defined so that, for any coherent sheaves $\mathcal{E}, \mathcal{F}$ on X,

$$(v(\mathcal{E}), v(\mathcal{F})) = -\chi(R\,\mathrm{Hom}(\mathcal{E}, \mathcal{F}))$$

Most of the following sections are adapted from the treatment in [KY00].

2.1. **Moduli of Sheaves.** Let H be an ample divisor on X, $v = (r, D, a) \in H^*(X, \mathbb{Z})$ a Mukai vector, and assume $v_1 = D$ is primitive. Throughout the following, by (semi)stability we will mean Gieseker (semi)stability with respect to H. Let $M(v)$ be the moduli space of semistable sheaves $\mathcal{E}$ with $v(\mathcal{E}) = v$. It is well known that (*e.g.*, see [HL])

Theorem 2.1. *For generic H, $M(v)$ is a proper smooth irreducible symplectic variety of dimension $2 + (v, v) = 2(g - ra)$ deformation equivalent to the Hilbert scheme of $g - ra$ points $X^{[g-ra]}$ on X.*

We will be concerned with the case when D is of minimal degree:

Definition 2.2. *A divisor class $D \in \mathrm{Pic}(X)$ has minimal degree if no positive line bundle has smaller intersection product with H, that is*

$$D.H = \min\{L.H | L \in \mathrm{Pic}(X), L.H > 0\}$$

Clearly every divisor of minimal degree is primitive. For any genus g, there is a suitable K3 surface with a divisor class D of genus g and minimal degree:

Examples 2.3. *(1) If X is an elliptic K3 surface with section, $\mathrm{Pic}(X) = \mathbb{Z}\sigma \oplus \mathbb{Z}f$, where f is the fiber class and σ the section class. Choosing $H = \sigma + 3f$ to be the ample class, we have*

$$(a\sigma + bf).H = a + b$$

σ and f are clearly of minimal degree, since both have intersection product 1 with H.

(2) If X has Picard rank one and H is the ample generator, then $D = H$ has minimal degree.

Lemma 2.4. *If $v_1 = D$ is of minimal degree with respect to H, then H is generic in the sense of* (2.1).

Proof. This follows from the fact the semistability implies stability. Indeed, for sheaves $\mathcal{E}$ with $v(\mathcal{E}) = (r, D, a)$, the minimal degree of D implies

(1) When $r > 0$, (semi)stability is equivalent to slope (semi)stability.
(2) When $r = 0$, stability is equivalent to purity.

in which case it's obvious that semistability implies stability. □

2.2. **A Stratification of the Moduli Spaces.** In the setup of Section (2.1) suppose further that $M(v)$ is a fine moduli space, so there exists a universal sheaf $\mathcal{F}$ on $X \times M(v)$, flat over $M(v)$, such that for every point $p = [\mathcal{E}] \in M(v)$, the restriction of $\mathcal{F}$ to $X \times p$ is $\mathcal{E}$. For our purposes we need only consider the case when the Euler characteristic $\chi = -(v(\mathcal{O}), v) \geq 0$ (*cf.*, (2.13)). Let $\pi : X \times M(v) \to M(v)$ be the projection, and consider the subsets,

$$M(v)_i = \{[\mathcal{E}] \in M(v) | \dim H^0(\mathcal{E}) = i\} \tag{5}$$

with the induced reduced subscheme structure. By the semicontinuity theorem, we have immediately

Lemma 2.5. *If $M(v)$ is a fine moduli space, then $\{M(v)_i\}_{i\geq 0}$ is a locally closed stratification of $M(v)$.*

In general $M(v)$ need not have a universal family, but étale locally it does. The cohomology of coherent sheaves can be computed étale locally, and closed and open immersions are both étale local properties, so

Proposition 2.6. *$\{M(v)_i\}_{i\geq 0}$ is a (finite) locally closed stratification of $M(v)$.*

The finiteness follows from the coherence of $\pi_*\mathcal{F}$ étale-locally.

Remark 2.7. *Since the second cohomology vanishes for any sheaf $\mathcal{E}$ with Mukai vector v,*

$$\dim H^0(\mathcal{E}) \geq \chi(\mathcal{E}) = \chi = r + a$$

The generic stratum is in fact $M(v)_{r+a}$, and each $M(v)_i$ for $0 \leq i < r + a$ is empty.

2.3. **Properties of Stable Pairs on K3 Surfaces.** Throughout this section, (semi)stability will mean Gieseker (semi)stability.

We briefly recall Le Potier's notion of a coherent system [LP93], henceforth referred to as a stable pair.

Definition 2.8. *A stable pair $(U, \mathcal{E})$ on X is a stable sheaf $\mathcal{E}$ and a subspace $U \subset$ $\mathrm{Hom}(\mathcal{O}, \mathcal{E})$. We will often denote a stable pair $(U, \mathcal{E})$ by the corresponding evaluation map $U \otimes \mathcal{O} \to \mathcal{E}$. A morphism $(U, \mathcal{E}) \to (U', \mathcal{E}')$ consists of morphisms $U \to U'$ and $\mathcal{E} \to \mathcal{E}'$ such that*

$$\begin{array}{ccc} U \otimes \mathcal{O} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ U' \otimes \mathcal{O} & \longrightarrow & \mathcal{E}' \end{array}$$

commutes. The Mukai vector of a stable pair $(U, \mathcal{E})$ is the Mukai vector of $\mathcal{E}$, and the level of $(U, \mathcal{E})$ is $\dim U$.

There is an obvious relative notion of stable pair. For a scheme S, let $\pi : X \times S \to S$ be the projection. A family of stable pairs $(\mathcal{U}, \mathcal{E})$ on $X \times S/S$ is a sheaf $\mathcal{E}$ on $X \times S$ flat over S, a locally free sheaf $\mathcal{U}$ on S, and a morphism $\pi^*\mathcal{U} \to \mathcal{E}$ such that the restriction to each fiber of π is a stable pair in the usual sense. A morphism of relative stable pairs $(\mathcal{U}, \mathcal{E})$ and $(\mathcal{U}', \mathcal{E}')$ is again given by morphisms $\mathcal{U} \to \mathcal{U}'$ and $\mathcal{E} \to \mathcal{E}'$ such that

$$\begin{array}{ccc} f^*\mathcal{U} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ f^*\mathcal{U}' & \longrightarrow & \mathcal{E}' \end{array}$$

commutes. By [LP93] the moduli functor of stable pairs with Mukai vector v and level n is (coarsely) representable by a projective scheme $\mathrm{Syst}^n(v)$, and the obvious forgetful morphism $p : \mathrm{Syst}^n(v) \to M(v)$ is projective.

The following lemma of Yoshioka will control the geometry of $\mathrm{Syst}^n(v)$:

Lemma 2.9. *Let D be a divisor on X of minimal degree, and $\mathcal{E}$ a stable sheaf on X with $\mathrm{c}_1(\mathcal{E}) = D$. Then*

(1) Given a subspace $U \subset \mathrm{Hom}(\mathcal{O}, \mathcal{E})$, let $\varphi : U \otimes \mathcal{O} \to \mathcal{E}$ be the evaluation map. Either
   (a) φ is injective,
$$0 \to U \otimes \mathcal{O} \to \mathcal{E} \to \mathcal{F} \to 0$$
   and the quotient $\mathcal{F}$ is stable.
   (b) φ is not injective,
$$0 \to \mathcal{F} \to U \otimes \mathcal{O} \to \mathcal{E} \to Q \to 0$$
   and the kernel is stable and locally free, while the quotient Q is dimension 0.

(2) Given $V \subset \mathrm{Ext}^1(\mathcal{E}, \mathcal{O})$, then in the corresponding extension
$$0 \to V^* \otimes \mathcal{O} \to \mathcal{F} \to \mathcal{E} \to 0$$
   $\mathcal{F}$ is stable.

Proof. See [Yos99, Lemma 2.1]. □

This has a number of geometric consequences. For example, we have

Theorem 2.10. *[KY00, Lemma 5.117]. Let X be a K3 surface, $v \in H^*(X, \mathbb{Z})$ a Mukai vector. For $v_1 = D$ of minimal degree, $\mathrm{Syst}^n(v)$ is smooth.*

Whenever $-(v(\mathcal{O}), v) \geq 0$, denote by $\mathrm{Syst}^n(v)_i$ the preimage of the stratum $M(v)_i$ from section (2.2) under the forgetful morphism $p : \mathrm{Syst}^n(v) \to M(v)$; clearly $\{\mathrm{Syst}^n(v)\}_{i \geq 0}$ is a locally closed stratification of $\mathrm{Syst}^n(v)$.

For $v = (r, D, a)$, denote $\mathrm{Syst}^n(r, D, a) = \mathrm{Syst}^n(v)$ and $M(r, D, a) = M(v)$. For $r \geq n$ there is a map (*cf.* [KY00])

$$q : \mathrm{Syst}^n(r, D, a) \to M(r - n, D, a - n)$$

mapping $(\mathcal{E}, U)$ to the cokernel $\mathcal{F}$ of the evaluation map $U \otimes \mathcal{O} \to \mathcal{E}$, which is injective by (2.9):

$$0 \to U \otimes \mathcal{O} \to \mathcal{E} \to \mathcal{F} \to 0$$

Again by (2.9) $\mathcal{F}$ is stable, and $v(\mathcal{F}) = v(\mathcal{E}) - v(\mathcal{O}^n) = (r - n, D, a - n)$ since $v(\mathcal{O}) = (1, 0, 1)$. Further, since $H^1(U \otimes \mathcal{O}) = 0$, the stratum $\mathrm{Syst}^n(r, D, a)_i$ maps into $M(r - n, D, a - n)_{i-n}$, assuming $r + a - 2n \geq 0$.

Lemma 2.11. *[KY00, Lemma 5.113]. For $-(v(\mathcal{O}), v) \geq 0$,*

1. *The restriction $\mathrm{Syst}^n(v)_i \to M(v)_i$ of the forgetful morphism p is an étale-locally trivial fibration with fiber $\mathrm{Gr}(n, i)$.*
2. *Furthermore, if $r + a \geq 2n$, then the restriction $\mathrm{Syst}^n(r, D, a)_i \to M(r - n, D, a - n)_{i-n}$ of the quotient morphism q is an étale-locally trivial fibration with fiber $\mathrm{Gr}(n, n + i - r - a)$.*

Proof. Both parts are obvious if $M(v)$ has a universal sheaf $\mathcal{F}$, in which case $\mathrm{Syst}^n(v)$ is a relative Grassmannian of $\mathcal{F}$. A universal sheaf exists étale locally, and the result follows. See [KY00]. □

The main tool for the computation of the Hodge polynomials of $\mathrm{Syst}^n(r, D, a)$ will be the existence of the resulting diagrams

$$\begin{array}{ccc} & \mathrm{Syst}^n(r, D, a)_i & \\ {}^{p}\swarrow & & \searrow^{q} \\ M(r, D, a)_i & & M(r - n, D, a - n)_{i-n} \end{array}$$

where p is an étale-local $\mathrm{Gr}(n, i)$-fibration and q is an étale local $\mathrm{Gr}(n, n + i - r - a)$-fibration.

One final property of the stable pair moduli spaces that will be relevant later is the duality, first proven by [Mar01, Theorem 39]

Proposition 2.12. *[KY00, Proposition 5.128]. In the setup of* (2.10) *there is an isomorphism*

$$\mathrm{Syst}^n(r, D, a) \cong \mathrm{Syst}^n(n - r, D, a - r)$$

for all $r \leq n$.

Remark 2.13. *By this duality, if we're interested in $\mathrm{Syst}^n(r, D, k + r)$ for $r \leq n$, we may assume $k \geq 0$, and thus we need only consider moduli spaces involving sheaves of nonnegative Euler characteristic.*

Proof of Proposition (2.12). We will at the very least define the map; see [KY00] for a proof of the theorem. Let $U \otimes \mathcal{O} \to \mathcal{E}$ be a stable pair, and consider $U \otimes \mathcal{O} \to \mathcal{E}$ as a morphism of complexes supported in degree 0 in the derived category $D^b(X)$; let $x \in D^b(X)$ be the cone. Thus, there is a triangle

$$x \to U \otimes \mathcal{O} \to \mathcal{E} \to x[1] \tag{6}$$

Alternatively, we can think of x as the 2-term complex $[U \otimes \mathcal{O} \to \mathcal{E}]$ with $\mathcal{E}$ placed in degree 1. Applying $R\mathcal{H}om(\,\cdot\,, \mathcal{O})$ to the triangle (6), we have a morphism

$$U^* \otimes \mathcal{O} \cong \mathcal{H}om(U \otimes \mathcal{O}, \mathcal{O}) \to \mathcal{H}om(x, \mathcal{O}) \tag{7}$$

One can show that $U^* \otimes \mathcal{O} \to \mathcal{H}om(x, \mathcal{O})$ is a stable pair and that this defines the isomorphism. For example, (7) is injective on global sections because, applying $R\,\mathrm{Hom}(\,\cdot\,, \mathcal{O})$ to (6), there is an exact sequence

$$0 \cong \mathrm{Hom}(\mathcal{E}, \mathcal{O}) \to U^* \to \mathrm{Hom}(x, \mathcal{O}) \to \mathrm{Ext}^1(\mathcal{E}, \mathcal{O}) \to 0 \tag{8}$$

where the triviality of $\mathrm{Hom}(\mathcal{E}, \mathcal{O})$ follows from the stability of $\mathcal{E}$. □

Remark 2.14. *In fact, by* (8)*, we obtain an isomorphism*

$$\mathrm{Syst}^n(r, D, a)_i \cong \mathrm{Syst}^n(n - r, D, a - r)_{i+n-\chi}$$

where $\chi = -(v(\mathcal{O}), v), v = (r, D, a)$.

3. Computation of Hodge polynomials

This section will be devoted to computing the generating functions of the moduli spaces of stable pairs on K3 surfaces. The geometric arguments are given here; some useful computations are collected in the subsequent section.

3.1. **Preparations.**

For X a scheme over $\mathbb{C}$, let

$$\mathrm{e}\,(X) = \sum_{p,q \geq 0} h^{p,q}(X)(-t)^p(-\bar{t})^q$$

denote the virtual Hodge polynomial of X. Throughout the following, we will set $u = t\bar{t}$; the Hodge polynomial of the Grassmannian $\mathrm{Gr}(k, n)$ of k planes in n-space is easily expressed in terms of u-integers (see Section (4.1)):

$$\mathrm{e}\,(\mathrm{Gr}(k, n)) = \begin{bmatrix} n \\ k \end{bmatrix}$$

In particular,

$$\mathrm{e}\,(\mathbb{P}^n) = [n + 1]$$

Let X now be a K3 surface. Recall that for a divisor class $D \in H^2(X, \mathbb{Z})$, $D^2 = 2g - 2$ by the adjunction formula, where g is the arithmetic genus of a divisor in the class D; g will be called the genus of D. For each genus $g \geq 0$ fix a polarized K3 surface X_g with polarization H_g and a divisor class D_g of minimal degree and genus g, *cf.* (2.3):

- $g = 0, 1$: $X_g \to \mathbb{P}^1$ is an elliptic K3 with a section. $\mathrm{Pic}(X_g) = \mathbb{Z}\sigma \oplus \mathbb{Z}f$, where f is the fiber class and σ the section class. For $g = 0$ take $H_0 = \sigma + 3f$ and $D_0 = \sigma$; for $g = 1$ take $H_1 = \sigma + 3f$ and $D_1 = f$.
- $g \geq 2$: X_g has Picard rank 1 with ample generator H_g of genus g; take $D_g = H_g$.

Denote by $M(r, D_g, k)$ the moduli space of H_g-stable rank r sheaves $\mathcal{E}$ on X_g with $\mathrm{c}_1(E) = D_g$ and $\mathrm{ch}_2(E).[X_g] = k$—in the notation of Section (2.1), this is $M(v)$ for $v(\mathcal{E}) = (r, D_g, k)$. Define infinite matrices $\mathbf{M(g)} = (M(g)_{ij})_{i,j\geq 0}$ and $\mathbf{Syst^n(g)} = (\mathrm{Syst}^n(g)_{ij})_{i,j\geq 0}$ of Hodge polynomials by

$$M(g)_{ij} = \begin{cases} \mathrm{e}\left(M\left(\frac{i-j}{2}, D_g, \frac{i+j}{2}\right)\right), & i-j \equiv 0 \mod 2 \\ 0, & i-j \equiv 1 \mod 2 \end{cases}$$

$$\mathrm{Syst}^n(g)_{ij} = \begin{cases} \mathrm{e}\left(\mathrm{Syst}^n\left(\frac{i-j}{2}, D_g, \frac{i+j}{2}\right)\right), & i-j \equiv 0 \mod 2 \\ 0, & i-j \equiv 1 \mod 2 \end{cases}$$

Note that these matrices only encode moduli of pairs $U \otimes \mathcal{O} \to \mathcal{E}$ for which $\mathcal{E}$ has nonnegative Euler characteristics, but by (2.13) this will be sufficient. Recall from Section 4.3 that in this case $M(r, D, a)_i$ is the stratum of $M(r, D, a)$ of sheaves $\mathcal{E}$ with $h^0(\mathcal{E}) = i$. By (2.7) the highest dimensional stratum is $i = r + a = \chi(\mathcal{E})$; define a matrix $\mathbf{M^0(g)} = (M^0(g)_{ij})_{i,j\geq 0}$ of the virtual Hodge polynomials of these generic strata:

$$M^0(g)_{ij} = \begin{cases} \mathrm{e}\left(M\left(\frac{i-j}{2}, D_g, \frac{i+j}{2}\right)_i\right), & i-j \equiv 0 \mod 2 \\ 0, & i-j \equiv 1 \mod 2 \end{cases}$$

3.2. **Encoding the Geometry.** In the following arguments, we will at any one time be considering $X = X_g$ for a fixed g, so we drop the g subscripts from the notation.

For any locally closed stratification of a scheme Y, the virtual Hodge polynomial of Y is the sum of the virtual Hodge polynomials of the strata. In particular,

$$\mathrm{e}\left(M(r, D, a)\right) = \sum_{i=0}^{\infty} \mathrm{e}\left(M(r, D, a)_i\right) \tag{9}$$

Of course the terms are zero for $i < \min(0, r + a)$. Similarly

$$\mathrm{e}\left(\mathrm{Syst}^n(r, D, a)\right) = \sum_{i=0}^{\infty} \mathrm{e}\left(\mathrm{Syst}^n(r, D, a)_i\right)$$

Recall from Section 4.5 that there is a diagram for $r \geq n$, $r + a \geq 2n$,

$$\begin{array}{ccc} & \mathrm{Syst}^n(r, D, a)_i & \\ \swarrow^{p} & & \searrow^{q} \\ M(r, D, a)_i & & M(r-n, D, a-n)_{i-n} \end{array}$$

which can be rewritten as

$$M(r+n,D,a+n)_{i+n} \xleftarrow{\;p\;} \mathrm{Syst}^n(r+n,D,a+n)_{i+n} \xrightarrow{\;q\;} M(r,D,a)_i$$

for $i, r, n \geq 0$. Recall that the fiber of p above $M(r+n,D,a+n)_{i+n}$ is $\mathrm{Gr}(n,i+n)$ and the fiber of q over $M(r,D,a)_i$ is $\mathrm{Gr}(n,i-r-a)$; we have $i \geq r+a$ since $h^0(\mathcal{E}) \geq \chi(\mathcal{E})$ for any stable $\mathcal{E}$ as $h^2(\mathcal{E}) = 0$. Taking $n = i-r-a$,

$$M(i-a,D,i-r)_{2i-r-a} \xleftarrow{\;p\;} \mathrm{Syst}^{i-r-a}(i-a,D,i-r)_{2i-r-a} \xrightarrow[\cong]{\;q\;} M(r,D,a)_i$$

where q is an isomorphism and p is an étale-locally fibration with fiber $\mathrm{Gr}(i-r-a, 2i-r-a)$.

For any Zariski-locally trivial fibration $Y \to S$ with fiber F—*i.e.*, Zariski-locally trivially on S, $Y \to S$ is isomorphic to the projection $F \times S \to S$—the Hodge polynomials simply multiply

$$\mathrm{e}\,(Y) = \mathrm{e}\,(F)\,\mathrm{e}\,(S)$$

The same is not in general true for étale-locally trivial fibrations, but it is in this case:

Lemma 3.1. *Let Y, S be quasiprojective varieties over $\mathbb{C}$, and $\pi : X \to Y$ a projective étale-locally trivial fibration with fiber $\mathrm{Gr}(k,n)$. Then*

$$\mathrm{e}\,(Y) = \mathrm{e}\,(\mathrm{Gr}(k,n))\,\mathrm{e}\,(S)$$

Proof. Let $\Omega = \Omega_{Y/S}$ be the relative cotangent bundle, and let $A \subset H^*_c(Y,\mathbb{Q})$ be the sub-Hodge structure generated by the Chern classes $\mathrm{c}_i(\Omega_{Y/S})$ and their products. For each fiber $i : \mathrm{Gr}(k,n) \to Y$, i^* clearly restricts to an isomorphism $A \xrightarrow{\cong} H^*(\mathrm{Gr}(k,n),\mathbb{Q})$ of Hodge structures. Let $\varphi : H^*(\mathrm{Gr}(k,n),\mathbb{Q}) \to A$ be the inverse, and define a morphism of Hodge structures

$$\psi = \varphi \smile \pi^* : H^*(\mathrm{Gr}(k,n),\mathbb{Q}) \otimes H^*_c(S,\mathbb{Q}) \to H^*_c(Y,\mathbb{Q})$$

By the Leray-Hirsch theorem, this is an isomorphism of vector spaces, and therefore of Hodge structures. □

Thus,

$$\begin{aligned} \mathrm{e}\,(M(r,D,a)_i) &= \mathrm{e}\,(\mathrm{Gr}(i-r-a, 2i-r-a))\,\mathrm{e}\,(M(i-a,D,i-r)_{2i-r-a}) \\ &= \begin{bmatrix} 2i-r-a \\ i-r-a \end{bmatrix} \mathrm{e}\,(M(i-a,D,i-r)_{2i-r-a}) \end{aligned} \tag{10}$$

After replacing $\ell = r$, $a = k + \ell$, and $i = k + 2\ell + s$, this becomes

$$\mathrm{e}\left(M(\ell, D, k+\ell)_{k+2\ell+s}\right) = \begin{bmatrix} k+2\ell+2s \\ s \end{bmatrix} \mathrm{e}\left(M(\ell+s, D, k+\ell+s)_{k+2\ell+2s}\right) \qquad (11)$$

The Hodge polynomial on the right is $M^0(g)_{k+2\ell+2s,k}$. The strata $M(\ell, D, k+\ell)_{k+2\ell+s}$ are null for $s < 0$, so

$$\begin{aligned} M(g)_{k+2\ell,k} &= \mathrm{e}\left(M(\ell, D, k+\ell)\right) \\ &= \sum_{s=0}^{\infty} \begin{bmatrix} k+2\ell+2s \\ s \end{bmatrix} M^0(g)_{k+2\ell+2s,k} \\ &= \sum_{s=0}^{\infty} A^0_{k+2\ell,k+2\ell+2s} M^0(g)_{k+2\ell+2s,k} \end{aligned}$$

where $\mathbf{A}(\mathbf{0}) = (A^0_{ij})_{i,j\geq 0}$ is the matrix from Section 4.4. Thus

$$\mathbf{M}(\mathbf{g}) = \mathbf{A}(\mathbf{0})\mathbf{M}^{\mathbf{0}}(\mathbf{g})$$

Moreover, since

$$\mathrm{e}\left(\mathrm{Syst}^n(r, D, a)_i\right) = \mathrm{e}\left(\mathrm{Gr}(n, i)\right) \mathrm{e}\left(M(r, D, a)_i\right)$$

We have

$$\mathrm{e}\left(\mathrm{Syst}^n(\ell, D, k+\ell)_{k+2\ell+s}\right) = \begin{bmatrix} k+2\ell+s \\ n \end{bmatrix} \mathrm{e}\left(M(\ell, D, k+\ell)_{k+2\ell+s}\right)$$

so that

$$\begin{aligned} \mathrm{Syst}^n(g)_{k+2\ell,k} &= \mathrm{e}\left(\mathrm{Syst}^n(\ell, D, k+\ell)\right) \\ &= \sum_{s=0}^{\infty} \begin{bmatrix} k+2\ell+s \\ n \end{bmatrix} \begin{bmatrix} k+2\ell+2s \\ s \end{bmatrix} M^0(g)_{k+2\ell+2s,k} \\ &= \sum_{s=0}^{\infty} A^n_{k+2\ell,k+2\ell+2s} M^0(g)_{k+2\ell+2s,k} \end{aligned}$$

where $\mathbf{A}(\mathbf{n}) = (A^n_{ij})_{i,j\geq 0}$ is the more general A-matrix from Section 4.4. Thus,

$$\mathbf{Syst}^{\mathbf{n}}(\mathbf{g}) = \mathbf{A}(\mathbf{n})\mathbf{M}^{\mathbf{0}}(\mathbf{g})$$

and setting $\mathbf{P}(\mathbf{n}) = \mathbf{A}(\mathbf{n})\mathbf{A}(\mathbf{0})^{-1}$,

Proposition 3.2.

$$\mathbf{Syst}^{\mathbf{n}}(\mathbf{g}) = \mathbf{P}(\mathbf{n})\mathbf{M}(\mathbf{g})$$

The entries of $\mathbf{P}(\mathbf{n})$ are computed in (4.5)

3.3. **Explicit Computations.** By (2.1), $M(r,D,a)$ is deformation equivalent to the Hilbert scheme of points $X^{[g-ra]}$, so

$$\mathrm{e}\left(M(r,D,a)\right) = \mathrm{e}\left(X^{[g-ra]}\right)$$

The generating function for the Hodge polynomials of the $X^{[n]}$ is, by Göttsche's formula [Göt90],

$$\begin{aligned}\sum_{n\geq 0}\mathrm{e}\left(X^{[n]}\right)q^n &= \prod_{n\geq 1}\prod_{i,j=0}^{2}(1-(-1)^{i+j}t^{i-1}\bar{t}^{j-1}(uq)^n)^{-(-1)^{i+j}h^{i,j}(X)}\\ &= \prod_{n\geq 1}\frac{1}{(1-u^{-1}(uq)^n)(1-t\bar{t}^{-1}(uq)^n)(1-(uq)^n)^{20}(1-\bar{t}t^{-1}(uq)^n)(1-u(uq)^n)}\end{aligned}$$

More concisely,

$$\begin{aligned}&\sum_{n\geq 0}\mathrm{e}\left(X^{[n]}\right)u^{-n}q^n =\\ &\quad=\prod_{n\geq 1}\frac{1}{(1-u^{-1}q^n)(1-t^2u^{-1}q^n)(1-q^n)^{20}(1-t^{-2}uq^n)(1-uq^n)}\end{aligned} \tag{12}$$

Denote by $c(n)=\mathrm{e}\left(X^{[n]}\right)$. We are interested in the generating function (we suppress the u-dependence from the notation)

$$\begin{aligned}F_n^r(q,y) :&= \sum_{g\geq 0}\sum_{k\in\mathbb{Z}}\mathrm{e}\left(\mathrm{Syst}^n(r,D_g,k+r)\right)u^{-g}y^kq^{g-1}\\ &= \sum_{g\geq 0}\sum_{k\geq 0}\mathrm{e}\left(\mathrm{Syst}^n(r,D_g,k+r)\right)u^{-g}y^kq^{g-1}\\ &\quad+\sum_{g\geq 0}\sum_{k<0}\mathrm{e}\left(\mathrm{Syst}^n(r,D_g,k+r)\right)u^{-g}y^kq^{g-1}\end{aligned} \tag{13}$$

The exponent $g-1$ of q (instead of simply g) is customary. For $r\leq n$, we know by (2.12) that

$$\mathrm{Syst}^n(r,D,r-k)\cong\mathrm{Syst}^n(n-r,D,n-r+k)$$

and therefore we can write (13) as

$$F_n^r(q,y) = \sum_{g\geq 0}\sum_{k\geq 0}\mathrm{Syst}^n(g)_{k+2r,k}u^{-g}y^kq^{g-1}+\sum_{g\geq 0}\sum_{k>0}\mathrm{Syst}^n(g)_{k+2(n-r),k}u^{-g}y^{-k}q^{g-1}$$

We have

$$\begin{aligned}
\mathrm{Syst}^n(g)_{k+2r,k} &= \sum_{\ell\geq r} P^n_{k+2r,k+2\ell} M(g)_{k+2\ell,k} \\
&= \sum_{\ell\geq r} P^n_{k+2r,k+2\ell} c\left(g-\ell^2-\ell k\right) \\
\mathrm{Syst}^n(g)_{k+2(n-r),k} &= \sum_{\ell\geq r+n} P^n_{k-2r+2n,k+2\ell} M(g)_{k+2\ell,k} \\
&= \sum_{\ell\geq r+n} P^n_{k-2r+2n,k+2\ell} c\left(g-\ell^2-\ell k\right)
\end{aligned}$$

Therefore

$$\begin{aligned}
F^r_n(q,y) &= \sum_{g\geq 0}\sum_{k\in\mathbb{Z}} u^{-g}q^{g-1}y^k\,\mathrm{Syst}^n(g)_{k+2r,k} \\
&= \sum_{g\geq 0}\sum_{k\geq 0} u^{-g}q^{g-1}y^k \sum_{\ell\geq r} P^n_{k+2r,k+2\ell} c(g-\ell^2-\ell k) \\
&\quad + \sum_{g\geq 0}\sum_{k\geq 1} u^{-g}q^{g-1}y^{-k} \sum_{\ell\geq n-r} P^n_{k-2r+2n,k+2\ell} c(g-\ell^2-\ell k)
\end{aligned}$$

and thus

$$F^r_n(q,y) = S(q)\sum_{k\geq 0}\sum_{\ell\geq r} y^k u^{-\ell^2-\ell k} q^{\ell k+\ell^2} P^n_{k+2r,k+2\ell} \tag{14}$$

$$+\, S(q)\sum_{k\geq 1}\sum_{\ell\geq n-r} y^{-k} u^{-\ell^2-\ell k} q^{\ell k+\ell^2} P^n_{k-2r+2n,k+2\ell} \tag{15}$$

where

$$S(q) = \sum_{g\geq 0} c(g) u^{-g} q^{g-1}$$

is the generating function of the Hodge polynomials of the Hilbert schemes of points on a K3 surface (again with the customary shift in the q power). We also know by (4.7) that

$$\begin{aligned}
P^n_{k+2r,k+2\ell} &= u^{r(n-r)} u^{\ell^2+\ell k-n\ell-kr} \frac{[k+2\ell]}{[n]} \begin{bmatrix} n+\ell-r-1 \\ n-1 \end{bmatrix} \begin{bmatrix} k+\ell+r-1 \\ n-1 \end{bmatrix} \\
P^n_{k-2r+2n,k+2\ell} &= u^{r(n-r)} u^{\ell^2+\ell k-n\ell-k(n-r)} \frac{[k+2\ell]}{[n]} \begin{bmatrix} \ell+r-1 \\ n-1 \end{bmatrix} \begin{bmatrix} k+\ell-r+n-1 \\ n-1 \end{bmatrix}
\end{aligned}$$

Note that the sums in (14), (15) make sense for all $\ell \geq 0$ since the terms are zero whenever $\ell < r$ in the first and $\ell < n-r$ in the second sum.

Write $p = \ell + k$ to get

$$\sum_{k\geq 0}\sum_{\ell\geq 0} y^k u^{-\ell^2-\ell k} q^{\ell k+\ell^2} P^n_{k+2r,k+2\ell}$$
$$= \frac{u^{r(n-r)}}{[n]} \sum_{\ell\geq 0}\sum_{p\geq \ell} u^{-n\ell-(p-\ell)r}[p+\ell]\begin{bmatrix} n+\ell-r-1 \\ n-1\end{bmatrix}\begin{bmatrix} p+r-1 \\ n-1\end{bmatrix} y^{p-\ell}q^{p\ell}$$

and

$$\sum_{k>0}\sum_{\ell\geq 0} y^{-k} u^{-\ell^2-\ell k} q^{\ell k+\ell^2} P^n_{k-2r+2n,k+2\ell}$$
$$= \frac{u^{r(n-r)}}{[n]} \sum_{p>\ell}\sum_{\ell\geq 0} u^{-np+(p-\ell)r}[p+\ell]\begin{bmatrix} \ell+r-1 \\ n-1\end{bmatrix}\begin{bmatrix} n-r+p-1 \\ n-1\end{bmatrix} y^{\ell-p}q^{p\ell}$$
$$\overset{\ell\leftrightarrow p}{=} \frac{u^{r(n-r)}}{[n]} \sum_{p\geq 0}\sum_{\ell> p} u^{-n\ell-(p-\ell)r}[p+\ell]\begin{bmatrix} n+\ell-r-1 \\ n-1\end{bmatrix}\begin{bmatrix} p+r-1 \\ n-1\end{bmatrix} y^{p-\ell}q^{p\ell}$$

Where the second line is obtained by setting $k = p - \ell$, and the third by switching p and ℓ. Noting that $\left[\begin{smallmatrix} n+\ell-r-1 \\ n-1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} p+r-1 \\ n-1\end{smallmatrix}\right]$ vanish for $\ell < r$ and $p < n-r$, respectively, the result is

Theorem 3.3. *For $r \leq n$,*

$$\frac{F^r_n(q,y)}{S(q)} = \frac{u^{r(n-r)}}{[n]} \sum_{\substack{p\geq n-r \\ \ell\geq r}} u^{-n\ell-(p-\ell)r}[p+\ell]\begin{bmatrix} n+\ell-r-1 \\ n-1\end{bmatrix}\begin{bmatrix} p+r-1 \\ n-1\end{bmatrix} y^{p-\ell}q^{p\ell}$$

Remark 3.4. *One is able to produce a similar formula for $r > n$ by once again using the duality* (2.12)*, but it requires defining $M(r, D, a)$ for negative r. Such moduli spaces naturally parametrize objects in the derived category $D^b(X)$ Verdier dual to stable sheaves; this will be further pursued in a subsequent paper by the authors.*

Remark 3.5. *The same method may be employed to compute the Hodge polynomials of the Brill-Noether strata $M(r, D_g, r + k)_i$ of each moduli space. For example, the generating function*

$$M^r_{\text{gen}}(q,y) = \sum_{g\geq 0}\sum_{k\geq 0} \mathrm{e}\left(M^0(r, D_g, k+r)\right) u^{-g}y^k q^{k-1}$$

(note that $k \geq 0$) can be computed by repeating the same calculation above, but with the $\mathbf{P}(\mathbf{n})$ matrix replaced with $\mathbf{B} = \mathbf{A}(\mathbf{0})^{-1}$ in (14) *(see Section* (4.4)*). Similarly,* (11) *can be used to obtain the generating function of the Hodge polynomials of the Brill-Nother strata of a fixed codimension.*

Note that the only dependence on $t, \bar{t}$ that doesn't factor through $u = t\bar{t}$ is from the term $S(q)$. In particular for $r = 0, n = 1$

$$\begin{aligned}\frac{F_1^0(q,y)}{S(q)} &= \sum_{\ell\geq 0}\sum_{p\geq 1} u^{-\ell}[p+\ell]y^{p-\ell}q^{p\ell}\\ &= \frac{1}{u-1}\sum_{\ell\geq 0}\sum_{p\geq 1}(u^p - u^{-\ell})y^{p-\ell}q^{p\ell}\\ &= \frac{1}{u-1}\Psi(u,y;q)\end{aligned}$$

where $\Psi(u, y; q)$ is the function from Section 4.3. By the computations in Section (4.3), we recover

Corollary 3.6. *[KY00, Theorem 5.158].*

$$\frac{F_1^0(q,y)}{S(q)} = \frac{-1}{(1-y)(1-u^{-1}y^{-1})}\prod_{n\geq 1}\frac{(1-q^n)^2(1-uq^n)(1-u^{-1}q^n)}{(1-yq^n)(1-y^{-1}q^n)(1-uyq^n)(1-u^{-1}y^{-1}q^n)}$$

For future reference, set

$$\Phi(u,y;q) = \prod_{n\geq 1}\frac{(1-q^n)^2(1-uq^n)(1-u^{-1}q^n)}{(1-yq^n)(1-y^{-1}q^n)(1-uyq^n)(1-u^{-1}y^{-1}q^n)}$$

Note directly from the formula in (3.3) that the duality (2.12) manifests itself in a kind of rank-level duality for the generating function $F_n^r(q, y)$:

Corollary 3.7.

$$F_n^r(q,y) = F_n^{n-r}(q,y^{-1})$$

3.4. **Relation to $r = 0, n = 1$.** The form of the higher generating functions is strongly determined by the Kawai-Yoshioka ($r = 0, n = 1$) function. Define Laurent polynomials $C_n^r(i, j)$ in u for $r \geq 0, n \geq 1$, $1 \leq i \leq n$ and $0 \leq j \leq n-i$ by $C_n^r(n, 0) = 1$ and

$$C_{n+1}^r(i,j) = C_n^r(i-1,j) + C_n^r(i+1,j-1) - u^{r-n}C_n^r(i,j-1) - u^{n-r}C_n^r(i,j)$$

Lemma 3.8. *The term* $u^{-n\ell-(p-\ell)r}[p+\ell]\begin{bmatrix} n+\ell-r-1 \\ n-1\end{bmatrix}\begin{bmatrix} p+r-1 \\ n-1\end{bmatrix}$ *is equal to*

$$\frac{(u-1)^{1-2n}}{[n-1]!^2}\sum_{i=1}^{n}\sum_{j=0}^{n-i}C_n^r(i,j)(u^{ip}-u^{-i\ell})u^{j(p-\ell)}$$

Proof. Clearly the claim is true for $n = 1$. Note that

$$\frac{u^{-\ell}[n+\ell-r][p+r-n]}{[n]^2} = \frac{(u-1)^2}{[n]^2}(u^{n-r} - u^{-\ell})(u^{p+r-n-\ell} - u^{\ell}) = \frac{(u-1)^2}{[n]^2}(u^p - u^{p+r-n-\ell} - u^{n-r} + u^{-\ell})$$

Thus by induction

$$u^{-(n+1)\ell-(p-\ell)r}[p+\ell]\begin{bmatrix} n+\ell-r-1 \\ n \end{bmatrix}\begin{bmatrix} p+r-1 \\ n \end{bmatrix} \tag{16}$$
$$= \frac{u^{-\ell}[n+\ell-r][p+r-n]}{[n]^2}\left(u^{-n\ell-(p-\ell)r}[p+\ell]\begin{bmatrix} n+\ell-r-1 \\ n-1 \end{bmatrix}\begin{bmatrix} p+r-1 \\ n-1 \end{bmatrix}\right)$$
$$= \frac{(u-1)^2}{[n]^2}(u^p - u^{p+r-n-\ell} - u^{n-r} + u^{-\ell})\left(\frac{(u-1)^{2-2n}}{[n-1]!^2}\sum_{i,j} C_n^r(i,j)(u^{ip} - u^{-i\ell})u^{j(p-\ell)}\right)$$

The two fractions match up to give the coefficient we want in front of the sum. Note that

$$(u^p + u^{-\ell})(u^{ip} - u^{-i\ell})u^{j(p-\ell)} = (u^{(i+1)p} - u^{p-i\ell})u^{j(p-\ell)} + (u^{ip-\ell} - u^{-(i+1)\ell})u^{j(p-\ell)} = (u^{(i+1)p} - u^{-(i+1)\ell})u^{j(p-\ell)} + (u^{(i-1)p} - u^{-(i-1)\ell})u^{(j+1)(p-\ell)}$$

and

$$-(u^{p+r-n-\ell} + u^{n-r})(u^{ip} - u^{-i\ell})u^{j(p-\ell)} = -u^{r-n}(u^{ip} - u^{-i\ell})u^{(j+1)(p-\ell)} - u^{n-r}(u^{ip} - u^{-i\ell})u^{j(p-\ell)}$$

So that in (16) the coefficient of $(u^{ip} - u^{-i\ell})u^{p-\ell}$ is

$$C_n^r(i-1,j) + C_n^r(i+1,j-1) - u^{r-n}C_n^r(i,j-1) - u^{n-r}C_n^r(i,j)$$

which by definition is $C_{n+1}^r(i,j)$.

□

By (3.3),

$$[n]u^{r(r-n)}S(q)^{-1}F_n^r(q,y) = \sum_{\ell\geq 0}\sum_{p\geq 0} u^{-n\ell-(p-\ell)r}[p+\ell]\begin{bmatrix} n+\ell-r-1 \\ n-1 \end{bmatrix}\begin{bmatrix} p+r-1 \\ n-1 \end{bmatrix} y^{p-\ell}q^{p\ell}$$
$$= \frac{(u-1)^{1-2n}}{[n-1]!^2}\sum_{i=1}^{n}\sum_{j=0}^{n-i} C_n^r(i,j)\sum_{p,\ell\geq 0}(u^{ip} - u^{-i\ell})u^{j(p-\ell)}y^{p-\ell}q^{p\ell}$$
$$= \frac{(u-1)^{1-2n}}{[n-1]!^2}\sum_{i=1}^{n}\sum_{j=0}^{n-i} C_n^r(i,j)\Psi(u^i, u^j y; q)$$

So finally

Theorem 3.9.

$$\frac{F_n^r(q,y)}{S(q)} = \frac{u^{r(n-r)}(u-1)^{1-2n}}{[n][n-1]!^2}\sum_{i=1}^{n}\sum_{j=0}^{n-i} C_n^r(i,j)\Psi(u^i,u^j y;q)$$

For example, for $n=2$ the only nonzero $C_2^r(i,j)$ are

$$C_2^r(2,0)=1 \quad C_2^r(1,0)=-u^{1-r} \quad C_2^r(1,1)=-u^{r-1}$$

and therefore

$$u^{r(r-2)}(u-1)^3[2]S(q)^{-1}F_2^r(q,y) = \Psi(u^2,y) - u^{1-r}\Psi(u,y) - u^{r-1}\Psi(u,uy) \qquad (17)$$

3.5. **Euler Characteristics and Modularity.** Of particular interest is the generating function $f_n^r(q,y) := F_n^r(q,y)|_{t=\bar{t}=1}$ of the Euler characteristics $\chi\left(\mathrm{Syst}^n(r,D,a)\right)$ of the stable pair moduli spaces. By definition,

$$f_n^r(q,y) = \sum_{g\geq 0}\sum_{k\in\mathbb{Z}} \chi\left(\mathrm{Syst}^n(r,D_g,k+r)\right) y^k q^{g-1}$$

The generating function $s(q)$ of the Euler characteristics of the Hilbert scheme of points is well known. From (12),

$$s(q) = S(q)|_{t=\bar{t}=1} = \sum_{g\geq 0}\chi(X^{[g]})q^{g-1} = q^{-1}\prod_{g\geq 1}\frac{1}{(1-q^g)^{24}} = \frac{1}{\eta(q)^{24}}$$

where $\eta(q)$ is the q-expansion of the Dedekind η function. Define

$$G_n^r(q,y) = \frac{F_n^r(q,y)}{S(q)}$$

and

$$g_n^r(q,y) = \frac{f_n^r(q,y)}{s(q)}$$

From (3.3),

Theorem 3.10.

$$g_n^r(q,y) = \frac{1}{n}\sum_{\substack{p\geq n-r\\ \ell\geq r}}(p+\ell)\binom{n+\ell-r-1}{n-1}\binom{p+r-1}{n-1}y^{p-\ell}q^{p\ell}$$

Note that the coefficient in (3.6) can be rewritten at $u=1$ as

$$\frac{-1}{(1-y)(1-y^{-1})} = \left(\sqrt{y}-\frac{1}{\sqrt{y}}\right)^{-2}$$

Thus, for $r=0$, $n=1$ we recover the Kawai-Yoshioka formula [KY00]

Corollary 3.11.

$$g_1^0(q,y) = \left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)^{-2} \prod_{n\geq 1} \frac{(1-q^n)^4}{(1-yq^n)^2(1-y^{-1}q^n)^2}$$

From [MPT] we know the v coefficients of $g_1^0(q,y)$ after the change of variable $y = -e^{iv}$ are (the q-expansions of) classical modular forms,

$$-g_1^0(q,-y) \overset{y=-e^{iv}}{=} \frac{1}{v^2} \cdot \exp\left(\sum_{g\geq 1} u^{2g} \frac{|B_{2g}|}{g\cdot(2g)!} E_{2g}(q)\right)$$

where $E_{2g}(q)$ is the q-expansion of the $2g$th Eisenstein series and B_{2g} is the $2g$th Bernoulli number, defined by $\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$. See [Fol09], for example, for an elementary treatment of modular forms. Note that

$$\frac{iv}{e^{iv}-1} = \sum_{m\geq 0} \frac{B_m (iv)^m}{m!}$$
$$\frac{-iv}{e^{-iv}-1} = \sum_{m\geq 0} \frac{B_m (-iv)^m}{m!}$$

thus

$$\begin{aligned}\lim_{u\to 1} \frac{v^2}{(1-u^{k+l}e^{iv})(1-u^{-l}e^{-iv})} &= \sum_{m,n\geq 0} B_m \frac{(iv)^m}{m!} B_n \frac{(-iv)^n}{n!} \\ &= \sum_{n\geq 0} \frac{i^n v^n}{n!} \sum_{k=0}^{n} (-1)^k B_k B_{n-k} \binom{n}{k}\end{aligned}$$

is a power series in $\mathbb{Q}[\![v]\!]$, which we denote by $\mathcal{B}$.

The divisor functions

$$\sigma_g(n) = \sum_{d|n} d^g$$

are related to the Eisenstein series by

$$E_{2g}(q) = 1 - \frac{4g}{B_{2g}} \sum_{n\geq 1} \sigma_{2g-1} q^n$$

$E_{2g}(q)$ is a modular form of weight $2g$ and level $\Gamma(1)$. The Eisenstein series $E_{2g+1}(q)$ of odd weight $2g+1$ and level $\Gamma(2)$ are defined by

$$E_{2g+1}(q) = 1 + \frac{4(-1)^g}{e_{2g}} \sum_{n\geq 1} \sigma_{2g-1} q^{n/2}$$

where the numbers e_n are defined by $\frac{1}{\cos t} = \sum_{n\geq 0} e_n \frac{t^n}{n!}$ Let

$$R = \mathbb{Q}(i)[E_{2g}(q), E_{2g+1}(q^2) | g \geq 1]$$

be an algebra generated by modular forms on $\Gamma(4)$. Clearly the generating functions $\Sigma_g = \sum_{n\geq 1} \sigma_g(n)q^n \in R$ for $g \geq 1$. The modularity result for $g_n^r(q,y)$ is:

Theorem 3.12. *The coefficient of v^s in the power series expansion of $v^2 g_n^r(q, e^{iv})$ is itself a power series in q, and this coefficient is in fact in the algebra R.*

First we have

Lemma 3.13. *Let $\log \Phi(u^k, u^\ell e^{iv}; q) = \sum_{s\geq 0} \psi_{k,\ell,s} v^s$ where $\psi_{k,\ell,s}$ is a function of u and q. Then for all $t \geq 0$, the t-th derivatives $\frac{d^t}{du^t}\psi_{k,\ell,s}|_{u=1} \in R$.*

Proof. By definition

$$\Phi(u^k, u^\ell e^{iv}, q) = \prod_{n\geq 1} \frac{(1-q^n)^2(1-u^kq^n)(1-u^{-k}q^n)}{(1-u^{k+\ell}e^{iv}q^n)(1-u^{-k-\ell}e^{-iv}q^n)(1-u^\ell e^{iv}q^n)(1-u^{-\ell}e^{-iv}q^n)}$$

and so

$$\begin{aligned}
\log \Phi(u^k, u^\ell e^{iv}; q) =& \sum_{n\geq 1} \big(2\log(1-q^n) + \log(1-u^kq^n) + \log(1-u^{-k}q^n) \\
& - \log(1-u^{k+\ell}e^{iv}q^n) + \log(1-u^{-k-\ell}e^{-iv}q^n) \\
& + \log(1-u^\ell e^{iv}q^n) + \log(1-u^{-\ell}e^{-iv}q^n)\big) \\
=& \sum_{n\geq 1}\sum_{r\geq 1} \frac{q^{nr}}{r}\left(2+u^{kr}+u^{-kr}\right) - \\
& \sum_{n\geq 1}\sum_{r\geq 1}\sum_{s\geq 0} \frac{q^{nr}(ivr)^s}{rs!}\left(u^{(k+\ell)r} + (-1)^s u^{-(k+\ell)r} + u^{\ell r} + (-1)^s u^{-\ell r}\right) \\
=& \sum_{n\geq 1} q^n \sum_{r|n} \frac{\left(2+u^{kr}+u^{-kr}\right)}{r} - \\
& \sum_{s\geq 0} \frac{i^s v^s}{s!} \sum_{n\geq 1} q^n \sum_{r|n} r^{s-1}\left(u^{(k+\ell)r} + (-1)^s u^{-(k+\ell)r} + u^{\ell r} + (-1)^s u^{-\ell r}\right)
\end{aligned}$$

This implies that

$$\psi_{k,\ell,0} = \sum_{n\geq 1} q^n \sum_{r|n} \frac{\left(2+u^{kr}+u^{-kr} - u^{(k+\ell)r} - u^{-(k+\ell)r} - u^{\ell r} - u^{-\ell r}\right)}{r}$$

and for $s \geq 1$

$$\psi_{k,\ell,s} = -\frac{i^s}{s!}\sum_{n\geq 1} q^n \sum_{r|n} r^{s-1}\left(u^{(k+\ell)r} + (-1)^s u^{-(k+\ell)r} + u^{\ell r} + (-1)^s u^{-\ell r}\right)$$

Evaluating at $u = 1$ we get $\psi_{k,\ell,0}|_{u=1} = 0$ and $\psi_{k,\ell,s}|_{u=1} = -\frac{2(1+(-1)^2)i^s}{s!}\Sigma_{s-1}$. Differentiating, we get that for $t \geq 1$ and $s \geq 1$ we have

$$\left(\frac{d^t}{du^t}\psi_{k,\ell,0}\right)\bigg|_{u=1} = t!\sum_{n\geq 1} q^n \sum_{r|n} r^{s-1}\left(\binom{kr}{t} + \binom{-kr}{t} - \binom{(k+\ell)r}{t} - \binom{-(k+\ell)r}{t} - \binom{\ell r}{t} - \binom{-\ell r}{t}\right)$$

and

$$\left(\frac{d^t}{du^t}\psi_{k,\ell,s}\right)\bigg|_{u=1} = -\frac{i^s t!}{s!}\sum_{n\geq 1} q^n \sum_{r|n} r^{s-1}\left(\binom{(k+\ell)r}{t} + (-1)^s\binom{-(k+\ell)r}{t} = + \binom{\ell r}{t} + (-1)^s\binom{-\ell r}{t}\right)$$

The conclusion then follows as each coefficient of q^n in the above expansions is either 0 or a $\mathbb{Q}$-linear combination of powers of r which implies that the derivative evaluated at $u = 1$ is a linear combination of terms of the form Σ_w for $w \geq 1$. □

Proof of Theorem 3.12. Note that

$$\begin{aligned} v^2 g^r_n(q, e^{iv}) &= \lim_{u\to 1} v^2 G^r_n(q, e^{iv}) \\ &= \lim_{u\to 1} \frac{u^{r(n-r)}(u-1)^{1-2n}}{[n][n-1]!^2} \sum_{k=1}^{n}\sum_{\ell=0}^{n-k} C^r_n(i,j)\Psi(u^k, u^\ell y; q) \\ &= \mathcal{B}\lim_{u\to 1} \frac{u^{r(n-r)}(u-1)^{1-2n}}{[n][n-1]!^2} \sum_{k=1}^{n}\sum_{\ell=0}^{n-k} C^r_n(i,j)\Phi(u^k, u^\ell y; q) \end{aligned}$$

To compute the limit we apply L'Hôpital observing that

$$\frac{d^{2n-1}}{du^{2n-1}}\frac{[n][n-1]!^2}{(u-1)^{2n-1}}\bigg|_{u=1} = n^2$$

We get

$$v^2 g^r_n(q, e^{iv}) = \frac{\mathcal{B}}{n^2}\frac{d^{n^2}}{du^{n^2}}\left(\sum_{k=1}^{n}\sum_{\ell=0}^{n-k} C^r_n(k,\ell)\Phi(u^k, u^\ell e^{iv}; q)\right)\bigg|_{u=1}$$

so it is enough to check that for all $t \geq 0$, $\frac{d^t}{du^t}\Phi(u^k, u^\ell e^{iv}; q)|_{u=1} \in R[\![v]\!]$.

But $\dfrac{d^t}{du^t}\Phi(u^k, u^\ell e^{iv}; q) = \dfrac{d^t}{du^t}\exp\left(\sum_{s\geq 0}\psi_{k,\ell,s}v^s\right)$ is of the form

$$\exp\left(\sum_{s\geq 0}\psi_{k,\ell,s}v^s\right)\mathcal{F}_{k,\ell,t} = \Phi(u^k, u^\ell e^{iv}; q)\mathcal{F}_{k,\ell,t}$$

where $\mathcal{F}_{k,\ell,t}$ is an expression involving only the $\psi_{k,\ell,s}$ and their derivatives. Evaluating at $u = 1$, the previous lemma shows that all coefficients of powers of v in $\mathcal{F}_{k,\ell,t}$ are in R. Finally, note that

$$\Phi(1, e^{iv}; q) = \prod_{n\geq 1}\frac{(1-q^n)^4}{(1-e^{iv}q^n)^2(1-e^{-iv}q^n)^2}$$

and this was computed in [MPT, p. 53] to be $4\sum_{k\geq 1}\dfrac{(-1)^k v^{2k}}{(2k)!}\Sigma_{2k-1}$. Multiplying everything together we get the required conclusion. □

4. Computations

4.1. **u-Binomial Coefficients.** The u-integer $[n]$ is the polynomial in u given by

$$[n] = \frac{u^n - 1}{u - 1}$$

The u-factorial and u-binomial coefficients are defined similarly:

$$[n]! = \prod_{s=1}^{n}[s] \qquad \begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \frac{[n]!}{[k]![n-k]!} & k \leq n \\ 0 & k > n \end{cases}$$

By fiat $[0]! = 1$.

4.2. **Properties of u-Binomial Coefficients.** Most binomial identities have u-analogs, many of which recover the classical identities in the $u \to 1$ limit. We collect here the properties we will need with proofs.

Lemma 4.1. *For any $k \leq n$*

(1)

$$[n] = [n-k] + u^{n-k}[k]$$

(2)

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} + u^{n+1-k}\begin{bmatrix} n \\ k-1 \end{bmatrix} \tag{18}$$

Proof. (1) Follows immediately from $[n+1] = \sum_{s=0}^{n} u^s$.

(2)

$$\begin{aligned}\begin{bmatrix} n+1 \\ k \end{bmatrix} &= \frac{[n+1]!}{[k]![n+1-k]!} \\ &= \frac{[n]!}{[k]![n-k]!}\left(\frac{[n+1]}{[n+1-k]}\right) \\ &= \frac{[n]!}{[k]![n-k]!}\left(1+u^{n+1-k}\frac{[k]}{[n+1-k]}\right) \\ &= \begin{bmatrix} n \\ k \end{bmatrix} + u^{n+1-k}\begin{bmatrix} n \\ k-1 \end{bmatrix}\end{aligned}$$

□

Note that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ has degree $k(n-k)$. The symmetric u-binomial coefficient is defined for $0 \le k \le n$ by

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = u^{-\frac{k(n-k)}{2}}\begin{bmatrix} n \\ k \end{bmatrix}$$

Also, under the same conditions let

$$\begin{Bmatrix} -n \\ k \end{Bmatrix} = (-1)^k\begin{Bmatrix} n+k-1 \\ k \end{Bmatrix}$$

Let

$$K_n(t,u) = \prod_{s=0}^{n-1}(1+tu^{s-\frac{n-1}{2}})$$

for $n \ge 0$.

Lemma 4.2.

$$K_n(t^{-1},u) = t^{-n}K_n(t,u)$$

Proof.

$$K_n(t^{-1},u) = t^{-n}\prod_{s=0}^{n-1}(t+u^{s-\frac{n-1}{2}})$$

but terms in the product come in pairs $(t+u^s)(t+u^{-s}) = (1+tu^s)(1+tu^{-s})$. □

K_n is invertible as a Laurent series in $t, u^{\frac{1}{2}}$; let

$$K_{-n}(t,u) = K_n(t,u)^{-1}$$

There is an analog of Lemma (4.1) for symmetric u-binomial coefficients:

Lemma 4.3. *For any* $0 \le k \le n$

(1)
$$\begin{Bmatrix} n+1 \\ k \end{Bmatrix} = u^{-\frac{k}{2}} \begin{Bmatrix} n \\ k \end{Bmatrix} + u^{\frac{n+1-k}{2}} \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \tag{19}$$

(2) $K_n(t,u)$ *is the generating function for the* $\begin{Bmatrix} n \\ k \end{Bmatrix}$, *that is*
$$K_n(t,u) = \sum_{k=0}^{\infty} t^k \begin{Bmatrix} n \\ k \end{Bmatrix}$$

(3)
$$\begin{Bmatrix} n+k \\ k \end{Bmatrix} = \sum_{s=0}^{k} u^{\frac{sn+s-k}{2}} \begin{Bmatrix} n+k-s-1 \\ k-s \end{Bmatrix}$$

(4) $K_{-n}(t,u)$ *is the generating function for the* $\begin{Bmatrix} -n \\ k \end{Bmatrix}$, *that is*
$$K_{-n}(t,u) = \sum_{k=0}^{\infty} t^k \begin{Bmatrix} -n \\ k \end{Bmatrix}$$

Proof. (1) Multiplying (18) by $u^{\frac{k(n+1-k)}{2}}$ gives (19).

(2) Note that
$$K_{n+1}(t,u) = \left(1 + tu^{\frac{n}{2}}\right) K_n(tu^{-\frac{1}{2}}, u) \tag{20}$$

Assuming by induction that the coefficient of t^s in $K_n(tu^{-\frac{1}{2}}, u)$ is $u^{-\frac{s}{2}} \begin{Bmatrix} n \\ s \end{Bmatrix}$, the coefficient of t^k in $K_{n+1}(t,u)$ is
$$u^{-\frac{k}{2}} \begin{Bmatrix} n \\ k \end{Bmatrix} + u^{\frac{n-k+1}{2}} \begin{Bmatrix} n \\ k-1 \end{Bmatrix}$$
which yields the result given part (1).

(3) Replacing n in (19) with $n+k-1$ we have
$$\begin{Bmatrix} n+k \\ k \end{Bmatrix} = u^{-\frac{k}{2}} \begin{Bmatrix} n+k-1 \\ k \end{Bmatrix} + u^{\frac{n}{2}} \begin{Bmatrix} n+k-1 \\ k-1 \end{Bmatrix} \tag{21}$$

Note that
$$\sum_{s=0}^{k} u^{\frac{sn+s-k}{2}} \begin{Bmatrix} n+k-s-1 \\ k-s \end{Bmatrix} = u^{-\frac{k}{2}} \begin{Bmatrix} n+k-1 \\ k \end{Bmatrix} + \sum_{s=1}^{k} u^{\frac{sn+s-k}{2}} \begin{Bmatrix} n+k-s-1 \\ k-s \end{Bmatrix}$$
$$= u^{-\frac{k}{2}} \begin{Bmatrix} n+k-1 \\ k \end{Bmatrix} + u^{\frac{n}{2}} \left(\sum_{s=0}^{k-1} u^{\frac{sn+s-k+1}{2}} \begin{Bmatrix} n+k-s-2 \\ k-s-1 \end{Bmatrix} \right)$$

By induction the term in parentheses is $\begin{Bmatrix} n+k-1 \\ k-1 \end{Bmatrix}$, and by (21) the result follows.

(4) Inverting (20), we have

$$K_{-n-1}(t,u) = \frac{1}{1+tu^{\frac{n}{2}}} K_{-n}(tu^{-\frac{1}{2}}, u) = K_{-n}(tu^{-\frac{1}{2}}, u) \sum_{s=0}^{\infty} (-1)^s t^s u^{\frac{ns}{2}}$$

Inductively assuming the coefficient of t^{k-s} in $K_{-n}(tu^{-\frac{1}{2}}, u)$ is

$$u^{-\frac{k-s}{2}} \begin{Bmatrix} -n \\ s \end{Bmatrix} = (-1)^{k-s} u^{-\frac{k-s}{2}} \begin{Bmatrix} n+k-s-1 \\ k-s \end{Bmatrix}$$

the coefficient of t^k in $K_{-n-1}(t,u)$ is

$$(-1)^k \sum_{s=0}^{k} u^{\frac{ns+s-k}{2}} \begin{Bmatrix} n+k-s-1 \\ k-s \end{Bmatrix} = (-1)^k \begin{Bmatrix} n+k \\ k \end{Bmatrix} = \begin{Bmatrix} -n-1 \\ k \end{Bmatrix}$$

by part (3).

□

4.3. q-**Theta Functions.** Given expressions a, b polynomial in q (we will be more precise below), the Pochhammer symbol $(a,b)_\infty$ is a formal power series in q defined by

$$(a,b)_\infty = \prod_{n=0}^{\infty} (1 - ab^n)$$

For example, $(q,q)_\infty = \prod_{n \geq 1} (1-q^n)$. The q-theta function $\Theta(x;q) \in \mathbb{Q}[x,x^{-1}][[q]]$ is a formal power series in q whose coefficients are Laurent polynomials in x. It is defined by

$$\Theta(x;q) = (q,q)_\infty (x,q)_\infty (x^{-1}q,q)_\infty = (1-x) \prod_{n=1}^{\infty} (1-q^n)(1-xq^n)(1-x^{-1}q^n)$$

In particular $\Theta(x;q)$ has a simple root at $x=1$. Our main use for $\Theta(x;q)$ is derived from an identity involving

$$\Phi(a,b;q) := \frac{(q,q)_\infty^3 \Theta(ab;q)}{\Theta(a;q)\Theta(b;q)}$$

Note $\Phi(a,b;q)$ is not an element of $\mathbb{Q}[a,b][\![q]\!]$, but it converges for $|q| < |a|, |b| < 1$. We have

Lemma 4.4. *For $n \in \mathbb{Z}$, define*

$$\operatorname{sign}(n) = \begin{cases} +1 & n \geq 0 \\ -1 & n < 0 \end{cases}$$

Then

$$\Phi(a,b;q)=\sum_{\mathrm{sign}(i)=\mathrm{sign}(j)}\mathrm{sign}(i)a^ib^jq^{ij}$$

for $|q|<|a|,|b|<1$*.*

Proof. See [Hic88, Theorem 1.5]. □

Define

$$\Psi(x,y;q)=\sum_{\ell\geq 0}\sum_{p\geq 1}(x^p-x^{-\ell})y^{p-\ell}q^{p\ell}$$

The actual statement we needed in (3.3) is

Lemma 4.5. *As formal power series*

$$\Psi(x,y;q)=\Phi(xy,y^{-1};q)$$

Proof. By [Hic88, Theorem 1.4],

$$\sum_{p\in\mathbb{Z}}\frac{a^p}{1-q^pb}=\Phi(a,b;q)$$

for $0<|q|<|a|<1$ and $b\neq q^p$ for any $p\in\mathbb{Z}$. On the region

$$R=\{(q,x,y)\in\mathbb{C}^3|0<|q|<|x|<|y^{-1}|<1\}$$

we have, for $p>0$, $|q^py|<1$, and for $p\geq 0$, $|q^py^{-1}|<1$. Thus, each line in the following converges in R:

$$\begin{aligned}
\Phi(xy,y^{-1};q)&=\sum_{p>0}\frac{(xy)^p}{1-q^py^{-1}}+\frac{1}{1-y^{-1}}+\sum_{p<0}\frac{(xy)^p}{1-q^py^{-1}}\\
&=\sum_{p>0}\frac{(xy)^p}{1-q^py^{-1}}+\frac{1}{1-y^{-1}}+\sum_{p>0}\frac{(xy)^{-p}}{1-q^{-p}y^{-1}}\\
&=\sum_{p>0}\frac{(xy)^p}{1-q^py^{-1}}-\frac{y}{1-y}-\sum_{p>0}\frac{(q^py)(xy)^{-p}}{1-q^py}\\
&=\sum_{p>0}\sum_{\ell\geq 0}(xy)^pq^{p\ell}y^{-\ell}-\sum_{p>0}\sum_{\ell\geq 0}(q^py)(xy)^{-p}q^{p\ell}y^{\ell}-\frac{y}{1-y}\\
&\overset{(*)}{=}\sum_{p>0}\sum_{\ell\geq 0}(xy)^pq^{p\ell}y^{-\ell}-\sum_{\ell>0}\sum_{p>0}(xy)^{-\ell}q^{p\ell}y^{p}-\frac{y}{1-y}\\
&=\sum_{p>0}\sum_{\ell>0}(xy)^pq^{p\ell}y^{-\ell}-\sum_{\ell>0}\sum_{p>0}(xy)^{-\ell}q^{p\ell}y^{p}+\frac{xy}{1-xy}-\frac{y}{1-y}\\
&=\sum_{p,\ell>0}(x^p-x^{-\ell})y^{p-\ell}q^{p\ell}+\frac{xy}{1-xy}-\frac{y}{1-y}
\end{aligned}$$

In the equality labeled (*) we replaced $\ell+1 \mapsto p$ and $p \mapsto \ell$. Thus, on R we have

$$\sum_{p,\ell>0}(x^p - x^{-\ell})y^{p-\ell}q^{p\ell} + \frac{xy}{1-xy} - \frac{y}{1-y} = \frac{(q,q)_\infty\Theta(x;q)}{\Theta(xy;q)\Theta(y^{-1};q)}$$
$$= \frac{(1-x)}{(1-xy)(1-y^{-1})}\prod_{n\geq 1}\frac{(1-q^n)^2(1-xq^n)(1-x^{-1}q^n)}{(1-xyq^n)(1-x^{-1}y^{-1}q^n)(1-yq^n)(1-y^{-1}q^n)}$$

which can be rewritten as

$$(1-xy)(1-y^{-1})\left(\sum_{p,\ell>0}(x^p - x^{-\ell})y^{p-\ell}q^{p\ell} + \frac{xy}{1-xy} - \frac{y}{1-y}\right)$$
$$= (1-x)\prod_{n\geq 1}\frac{(1-q^n)^2(1-xq^n)(1-x^{-1}q^n)}{(1-xyq^n)(1-x^{-1}y^{-1}q^n)(1-yq^n)(1-y^{-1}q^n)} \tag{22}$$

For any x, y with $|x| < |y^{-1}|$, (22) is an equality of series in $\mathbb{C}[\![q]\!]$ converging for $|q| < |x|$. Therefore it must be an equality of formal power series in $\mathbb{C}[x, y, x^{-1}, y^{-1}][\![q]\!]$. Since both sides converge for $|q|, |xy|, |y| < 1$ it follows it must be an equality of series in $\mathbb{C}[\![q]\!]$ for any such x, y; therefore, in that case, it must be that $(1-xy)(1-y^{-1})\sum_{p>0,\ell\geq 0}(x^p - x^{-\ell})y^{p-\ell}q^{p\ell}$ is equal to

$$(1-x)\prod_{n\geq 1}\frac{(1-q^n)^2(1-xq^n)(1-x^{-1}q^n)}{(1-xyq^n)(1-x^{-1}y^{-1}q^n)(1-yq^n)(1-y^{-1}q^n)}$$

and the conclusion follows. □

4.4. **A Useful Matrix.** In Section (3.2) we used the matrix $\mathbf{A}(\mathbf{n}) = (A^n_{ij})_{i,j\geq 0}$ defined by

$$A^n_{ij} = \begin{cases} \begin{bmatrix} \frac{i+j}{2} \\ n \end{bmatrix}\begin{bmatrix} j \\ \frac{j-i}{2} \end{bmatrix} & i-j \equiv 0 \mod 2 \\ 0 & i-j \equiv 1 \mod 2 \end{cases}$$

i.e., the only nonzero entries are $A^n_{k,k+2\ell} = \begin{bmatrix} k+\ell \\ n \end{bmatrix}\begin{bmatrix} k+2\ell \\ \ell \end{bmatrix}$, $k, \ell \geq 0$. In particular, $A^0_{k,k+2\ell} = \begin{bmatrix} k+2\ell \\ \ell \end{bmatrix}$. $\mathbf{A}(\mathbf{0})$ is upper triangular with ones along the diagonal, and is therefore invertible:

Proposition 4.6. *The inverse of* $\mathbf{A}(\mathbf{0})$ *is the matrix* $\mathbf{B} = (B_{ij})_{i,j\geq 0}$ *given by*

$$B_{k,k+2\ell} = (-1)^\ell u^{\binom{\ell}{2}}\frac{[k+2\ell]}{[k+\ell]}\begin{bmatrix} k+\ell \\ \ell \end{bmatrix}$$

and $B_{k,k+2\ell+1} = 0$, *for* $k, \ell \geq 0$

Proof. We need only check that the $(k, k+2\ell)$ entry of $\mathbf{A}(\mathbf{0})\mathbf{B}$ for $\ell > 0$ is 0, since the diagonal terms are clearly 1 and both matrices are upper triangular. The relevant entries of $\mathbf{B}$ are

$$B_{k+2s,k+2\ell} = (-1)^{\ell-s} u^{\binom{\ell-s}{2}} \begin{bmatrix} k+\ell+s \\ \ell-s \end{bmatrix} \frac{[k+2\ell]}{[k+\ell+s]}$$

Also note that

$$\begin{aligned} \begin{bmatrix} k+2s \\ s \end{bmatrix} \begin{bmatrix} k+\ell+s \\ \ell-s \end{bmatrix} \frac{[k+2\ell]}{[k+\ell+s]} &= \left(\frac{[k+2s]!}{[s]![k+s]!} \right) \left(\frac{[k+s+\ell]!}{[\ell-s]![k+2s]!} \right) \frac{[k+2\ell]}{[k+\ell+s]} \\ &= \left(\frac{[\ell]!}{[s]![\ell-s]!} \right) \left(\frac{[k+s+\ell-1]!}{[k+s]![\ell-1]!} \right) \frac{[k+2\ell]}{[\ell]} \\ &= \begin{bmatrix} \ell \\ s \end{bmatrix} \begin{bmatrix} k+\ell-1 \\ \ell-1 \end{bmatrix} \frac{[k+2\ell]}{[\ell]} \end{aligned}$$

Thus

$$\begin{aligned} \sum_{s=0}^{\infty} A^0_{k,k+2s} B_{k+2s,k+2\ell} &= \sum_{s=0}^{\ell} (-1)^{\ell-s} u^{\binom{\ell-s}{2}} \begin{bmatrix} k+2s \\ s \end{bmatrix} \begin{bmatrix} k+\ell+s \\ \ell-s \end{bmatrix} \frac{[k+2\ell]}{[k+\ell+s]} \\ &= \left(\frac{[k+2\ell]}{[\ell]} \right) \sum_{s=0}^{\ell} (-1)^{\ell-s} u^{\binom{\ell-s}{2}} \begin{bmatrix} k+s+\ell-1 \\ \ell-1 \end{bmatrix} \begin{bmatrix} \ell \\ s \end{bmatrix} \\ &= \left(\frac{[k+2\ell]}{[\ell]} \right) \sum_{s=0}^{\ell} (-1)^{\ell-s} u^{\binom{\ell-s}{2} + \frac{(\ell-1)(k+s)}{2} + \frac{s(\ell-s)}{2}} \begin{Bmatrix} k+s+\ell-1 \\ \ell-1 \end{Bmatrix} \begin{Bmatrix} \ell \\ s \end{Bmatrix} \\ &= u^{\frac{\ell^2-\ell+(\ell-1)k}{2}} \left(\frac{[k+2\ell]}{[\ell]} \right) \sum_{s=0}^{\ell} (-1)^{\ell-s} \begin{Bmatrix} k+s+\ell-1 \\ \ell-1 \end{Bmatrix} \begin{Bmatrix} \ell \\ s \end{Bmatrix} \\ &= (-1)^{k+\ell} u^{\frac{\ell^2-\ell+(\ell-1)k}{2}} \left(\frac{[k+2\ell]}{[\ell]} \right) \sum_{s=0}^{\ell} \begin{Bmatrix} -\ell \\ k+s \end{Bmatrix} \begin{Bmatrix} \ell \\ s \end{Bmatrix} \end{aligned}$$

By (4) of (4.3), $\begin{Bmatrix} -\ell \\ k+s \end{Bmatrix}$ is the coefficient of t^{k+s} in $K_{-\ell}(t,q)$ and $\begin{Bmatrix} \ell \\ s \end{Bmatrix}$ is the coefficient of t^{-s} in $K_\ell(t^{-1},q)$. Therefore, the sum is the coefficient of t^k in $K_{-\ell}(t,q)K_\ell(t^{-1},q) = t^{-\ell}K_{-\ell}(t,q)K_\ell(t,q) = t^{-\ell}$ so it must be 0, unless $\ell = k = 0$, but we assumed $\ell > 0$. □

4.5. **A Useful Product.** In Section (3.3), an explicit computation of the product $\mathbf{P}(\mathbf{n}) := \mathbf{A}(\mathbf{n})\mathbf{A}(\mathbf{0})^{-1}$ enabled us to perform the calculation. The product matrix $\mathbf{P}(\mathbf{n}) = (P^n_{ij})_{i,j\geq 0}$ is given by

Lemma 4.7. *For $k, \ell \geq 0$, $n > 0$,*

$$P^n_{k,k+2\ell} = u^{\ell^2+\ell(k-n)} \frac{[k+2\ell]}{[n+\ell]} \begin{bmatrix} n+\ell \\ n \end{bmatrix} \begin{bmatrix} k+\ell-1 \\ n-1 \end{bmatrix}$$

and $P^n_{k,k+2\ell+1} = 0$.

Proof. The proof is a calculation very similar to the proof of lemma (4.3). Note that for $\ell \geq s$

$$\begin{aligned}
&\begin{bmatrix} k+s \\ n \end{bmatrix} \begin{bmatrix} k+2s \\ s \end{bmatrix} \begin{bmatrix} k+s+\ell \\ \ell-s \end{bmatrix} = \\
&\qquad = \frac{[k+s]\cdots[k+s-n+1]}{[n]!} \frac{[k+2s]\cdots[k+s+1]}{[s]!} \frac{[k+s+\ell]\cdots[k+2s+1]}{[\ell-s]!} \\
&\qquad = \frac{[k+s+\ell]!}{[n]![s]![\ell-s]![k+s-n]!} \\
&\qquad = \left(\frac{[n+\ell]!}{[n]![\ell]!}\right) \left(\frac{[\ell]!}{[s]![\ell-s]!}\right) \left(\frac{[k+s+\ell-1]!}{[k+s-n]![n+\ell-1]!}\right) \frac{[k+s+\ell]}{[n+\ell]}
\end{aligned}$$

so

$$\begin{aligned}
P^n_{k,k+2\ell} &= \sum_{s=0}^{\ell} A^n_{k,k+2s} B_{k+2s,k+2\ell} \\
&= \sum_{s=0}^{\ell} (-1)^{\ell-s} u^{\binom{\ell-s}{2}} \begin{bmatrix} k+s \\ n \end{bmatrix} \begin{bmatrix} k+2s \\ s \end{bmatrix} \begin{bmatrix} k+s+\ell \\ \ell-s \end{bmatrix} \frac{[k+2\ell]}{[k+s+\ell]} \\
&= \frac{[k+2\ell]}{[n+\ell]} \begin{bmatrix} n+\ell \\ n \end{bmatrix} \sum_{s=0}^{\ell} (-1)^{\ell-s} u^{\binom{\ell-s}{2}} \begin{bmatrix} k+s+\ell-1 \\ n+\ell-1 \end{bmatrix} \begin{bmatrix} \ell \\ s \end{bmatrix} \\
&= \frac{[k+2\ell]}{[n+\ell]} \begin{bmatrix} n+\ell \\ n \end{bmatrix} \sum_{s=0}^{\ell} (-1)^{\ell-s} u^{\binom{\ell-s}{2} + \frac{(n+\ell-1)(k-n+s)}{2} + \frac{s(\ell-s)}{2}} \begin{Bmatrix} k+s+\ell-1 \\ n+\ell-1 \end{Bmatrix} \begin{Bmatrix} \ell \\ s \end{Bmatrix} \\
&= \frac{[k+2\ell]}{[n+\ell]} \begin{bmatrix} n+\ell \\ n \end{bmatrix} u^{\frac{\ell^2-\ell+(n+\ell-1)(k-n)}{2}} \sum_{s=0}^{\ell} (-1)^{\ell-s} u^{sn/2} \begin{Bmatrix} k+s+\ell-1 \\ n+\ell-1 \end{Bmatrix} \begin{Bmatrix} \ell \\ s \end{Bmatrix} \\
&= (-1)^{k-n+\ell} \frac{[k+2\ell]}{[n+\ell]} \begin{bmatrix} n+\ell \\ n \end{bmatrix} u^{\frac{\ell^2-\ell+(n+\ell-1)(k-n)}{2}} \sum_{s=0}^{\ell} u^{sn/2} \begin{Bmatrix} -(n+\ell) \\ k-n+s \end{Bmatrix} \begin{Bmatrix} \ell \\ s \end{Bmatrix} \quad (23)
\end{aligned}$$

$u^{sn/2} \begin{Bmatrix} -(n+\ell) \\ k-n+s \end{Bmatrix}$ is the coefficient of t^{k-n+s} in $u^{(n^2-kn)/2} K_{-(n+\ell)}(tu^{n/2}, u)$ and $\begin{Bmatrix} \ell \\ s \end{Bmatrix}$ is the coefficient of t^{-s} in $K_\ell(t^{-1}, u)$. Therefore, the sum in (23) is the coefficient of

t^{k-n} in

$$\begin{aligned} u^{(n^2-kn)/2}K_{-(n+\ell)}(tu^{n/2},u)K_\ell(t^{-1},u) &= u^{(n^2-kn)/2}t^{-\ell}K_{-(n+\ell)}(tu^{n/2},u)K_\ell(t,u) \\ &= u^{(n^2-kn)/2}t^{-\ell}K_{-n}(tu^{(n+\ell)/2},u) \end{aligned}$$

which is

$$\begin{aligned} u^{\frac{\ell^2+\ell k}{2}}\begin{Bmatrix} -n \\ k-n+\ell \end{Bmatrix} &= (-1)^{k-n+\ell}u^{\frac{\ell^2+\ell k}{2}}\begin{Bmatrix} k+\ell-1 \\ n-1 \end{Bmatrix} \\ &= (-1)^{k-n+\ell}u^{\frac{\ell^2+\ell k-(n-1)(k+\ell-n)}{2}}\begin{bmatrix} k+\ell-1 \\ n-1 \end{bmatrix} \end{aligned}$$

and we get

$$P^n_{k,k+2\ell} = u^{\ell^2+\ell k-n\ell}\frac{[k+2\ell]}{[n+\ell]}\begin{bmatrix} n+\ell \\ n \end{bmatrix}\begin{bmatrix} k+\ell-1 \\ n-1 \end{bmatrix}$$

□

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