A SHORT PROOF OF A CONJECTURE OF MATSUSHITA

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Abstract. In this note we build on the arguments of van Geemen and Voisin [17] to prove a conjecture of Matsushita that a Lagrangian fibration of an irreducible hyperkähler manifold is either isotrivial or of maximal variation. We also complete a partial result of Voisin [19] regarding the density of torsion points of sections of Lagrangian fibrations.

Let X be an irreducible compact hyperkähler manifold, that is, a simply-connected compact Kähler manifold X for which $H^0(X,\Omega^2_X) = \mathbb{C}\sigma$ for a nowhere-degenerate holomorphic two-form σ. A Lagrangian fibration of X is a proper morphism $f : X \to B$ to a normal compact analytic variety B whose generic fiber is smooth, connected, and Lagrangian (see [10] for a recent survey). It follows that every smooth fiber is an abelian variety. We let $B^\circ \subset B$ be a dense Zariski open smooth subset over which the restriction $f^\circ : X^\circ \to B^\circ$ is smooth. By the period map of f we mean the period map $\varphi : B^\circ \to S$ to an appropriate moduli space S of polarized abelian varieties associated to the natural variation of (polarized) weight one integral Hodge structures on B° with underlying local system $R^1f^\circ_*\mathbb{Z}_{X^\circ}$. We say f is *isotrivial* if the period map is trivial (equivalently if $R^1f^\circ_*\mathbb{Z}_{X^\circ}$ has finite monodromy) and *of maximal variation* if the period map is generically finite.

Our main result is to resolve a conjecture of Matsushita:

Theorem 1. *Let X be an irreducible hyperkähler manifold (or more generally a primitive symplectic variety in the sense of* [2]*). Then any Lagrangian fibration $f : X \to B$ is either isotrivial or of maximal variation.*

Both possibilities in Theorem 1 occur, even for K3 surfaces—see for example [9, Chapter 11]. Primitive symplectic varieties are the natural singular analog (as far as deformation theory is concerned) of irreducible hyperkähler manifolds; see below for the definition and the precise meaning of a Lagrangian fibration in this context. Let $T_0 \subset H^2(X,\mathbb{Q})$ be the rational transcendental lattice, namely, the smallest rational Hodge substructure containing $[\sigma] \in H^{2,0}(X)$. Theorem 1 was proven by van Geemen and Voisin [17, Theorem 5] assuming X is smooth and projective, that T_0 has generic (special) Mumford–Tate group (namely $\mathrm{SO}(T_0, q_X)$, where q_X is the Beauville–Bogomolov–Fujiki form), and that $\operatorname{rk} T_0 \geq 5$, by showing that under these conditions any fiber of a Lagrangian fibration that is not of maximal variation must be a factor of the Kuga–Satake variety of T_0. Their result in particular applies to the generic projective deformation of $f : X \to B$, at least for X of a known deformation type.

We will instead prove Theorem 1 by considering the complex variation of Hodge structures on $R^1f^\circ_*\mathbb{C}_{X^\circ}$. We first recall the basic properties of complex variations. A complex variation of Hodge structures on a Zariski open subset of a compact analytic variety (see for example [5]) consists of a $\mathbb{C}$-local system V and a holomorphic (resp. antiholomorphic) descending filtration $F^\bullet$ (resp. $\overline{F}^\bullet$) such that we have a splitting of the sheaf of C^∞ sections $A^0(V) = \bigoplus_p A^0(V^p)$ where $V^p = F^p \cap \overline{F}^{-p}$ and the flat connection maps $A^0(V^p)$ to $A^{1,0}(V^{p-1}) \oplus A^1(V^p) \oplus A^{0,1}(V^{p+1})$. We refer to the grading V^p as the Hodge grading and we say the level of the variation is the difference $p_{\max} - p_{\min}$ where $p_{\max}$ (resp. $p_{\min}$) is the maximum (resp. minimum) Hodge degree p for which $V^p \neq 0$. Observe that the level of a tensor product $V \otimes W$ is the sum of the levels of V and W. A polarization of the variation is a flat hermitian form h for which the splitting is orthogonal and $(-1)^p h$ is positive definite on V^p. In this case $\overline{F}^{-p} = (F^{p+1})^{\perp}$. A variation which admits a polarization is said to be polarizable. We define $\mathbb{C}(-d)$ to be the polarizable complex Hodge structure on $V = \mathbb{C}$ with $V^d = V$.

Recall that the category of polarizable complex variations of Hodge structures is semi-simple. The theorem of the fixed part [16] says that for two polarizable complex variations V, W, the group $\mathrm{Hom}(V, W)$ of morphisms of local systems has a natural complex Hodge structure whose degree zero part is exactly the morphisms of complex variations. We have the following further consequence due to Deligne:

Theorem 2 ([5, 1.13 Proposition]). *Suppose V is a $\mathbb{C}$-local system underlying a polarizable complex variation of Hodge structures and that we have a splitting of $\mathbb{C}$-local systems*

$$V = \bigoplus_i M_i \otimes A_i \tag{1}$$

where the M_i are irreducible and pairwise non-isomorphic and the A_i are nonzero complex vector spaces. Then

(1) Each M_i underlies a polarizable complex variation of Hodge structures, unique up to shifting the Hodge grading.
(2) Each polarizable complex variation of Hodge structures with underlying local system V arises from (1) by equipping each M_i with its unique polarizable complex variation of Hodge structures and each A_i with a uniquely determined polarizable complex Hodge structure (up to shifting the Hodge grading), namely $A_i = \mathrm{Hom}(M_i, V)$.

In particular, the theorem implies a polarizable complex variation is irreducible if and only if the underlying local system is.

Given an $\mathbb{R}$-local system V, a polarizable real variation of Hodge structures[1] on V in the usual sense naturally induces a polarizable complex variation on $V_{\mathbb{C}}$. Conversely, a polarizable complex variation on $V_{\mathbb{C}}$ comes from a polarizable real variation on V if complex conjugation flips the Hodge grading, or more precisely if for some (hence any) polarization h the isomorphism of local systems $V_{\mathbb{C}} \to V_{\mathbb{C}}^{\vee}$ given by $y \mapsto h(-, \overline{y})$ induces an isomorphism of complex variations $V_{\mathbb{C}} \to V_{\mathbb{C}}^{\vee}(-w)$ for some (uniquely determined) w. Indeed, if this is the case then $V^p \xrightarrow{\cong} (V^{w-p})^{\vee}$ so $\overline{V^p} = V^{w-p}$. Moreover, for even w (resp. odd w) a real polarization is provided by the symmetric (resp. antisymmetric) real form $q(x, y) = h(x, \overline{y}) + h(y, \overline{x})$ (resp. $q(x, y) = i(h(x, \overline{y}) - h(y, \overline{x}))$), since $q(x, \overline{x}) = h(x, x) + h(\overline{x}, \overline{x})$ (resp. $-iq(x, \overline{x}) = h(x, x) - h(\overline{x}, \overline{x})$) is definite of alternating sign on V^p.

The category of polarizable real variations is also semi-simple. Observe that by Theorem 2 any isotypic component W of a polarizable real variation V is a real subvariation, as the same is true over $\mathbb{C}$ and the isomorphism $V_{\mathbb{C}} \to V_{\mathbb{C}}^{\vee}(-w)$ coming from h restricts to an isomorphism $W_{\mathbb{C}} \to W_{\mathbb{C}}^{\vee}(-w)$. If V is a single isotypic factor, then $V_{\mathbb{C}}$ either has one self-conjugate irreducible factor N or has two non-isomorphic conjugate irreducible factors $N, \overline{N}$. Note that $N^{\vee} \cong \overline{N}$ via the polarization, and that the level of V is at least as large as the level of any of the irreducible factors of $V_{\mathbb{C}}$.

[1] Throughout, by a real variation we mean a pure real variation, unless otherwise specified.

We say that a real or complex variation is *isotrivial* if the Hodge filtration is flat, or equivalently if the irreducible factors of the complexification are level zero[2]. To summarize the above discussion:

Lemma 3. *Let V be an irreducible polarizable real variation of Hodge structures of level one. Then V is either isotrivial, or every irreducible factor of $V_{\mathbb{C}}$ is level one.*

Before turning to the proof of Theorem 1 we recall the definition of a primitive symplectic variety. Let X be a symplectic variety in the sense of Beauville[3] [3], that is, a compact Kähler variety with rational singularities and a nowhere degenerate 2-form σ on its regular locus X^{reg}. We say that X is primitive symplectic if $H^1(X, \mathcal{O}_X) = 0$ and $H^0(X^{\mathrm{reg}}, \Omega^2_{X^{\mathrm{reg}}}) = \mathbb{C}\sigma$. As the singularities are rational, for any resolution $\pi : Y \to X$ the form σ extends to a two-form on Y [11, Corollary 1.7]. Moreover, $\pi^* : H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$ is injective, so the Hodge structure on $H^2(X, \mathbb{Q})$ is pure, and we have an induced isomorphism $\pi^* : H^{2,0}(X) \to H^{2,0}(Y)$ (see [2] for details). In particular we have a well defined class $[\sigma] \in H^{2,0}(X)$.

By a Lagrangian fibration of a primitive symplectic variety we still mean a proper morphism $f : X \to B$ to a normal compact analytic variety B whose generic fiber is smooth, connected, and Lagrangian. Each smooth fiber will still be an abelian variety. Moreover, B is in fact Kähler and Moishezon [18] (and in particular an algebraic space) since f is equidimensional as in [10, Lemma 1.17], now using functorial pullback of reflexive forms [11, Theoreom 1.11] and the fact that $R\pi_*\omega_Y \cong \omega_X \cong \mathcal{O}_X$ by the rationality of the singularities of X [12, §5.1].

We use the same notation as above: $B^\circ \subset B$ is a dense Zariski open smooth subset over which the restriction $f^\circ : X^\circ \to B^\circ$ is smooth and $\varphi : B^\circ \to S$ is the period map associated to the variation of (polarized) weight one integral Hodge structures on B° with underlying local system $R^1 f^\circ_* \mathbb{Z}_{X^\circ}$.

Proof of Theorem 1. Let $V_{\mathbb{Z}} := R^1 f^\circ_* \mathbb{Z}_{X^\circ}$. We start with the following result of Voisin, whose proof we give for convenience (and to extend it slightly).

Lemma 4 ([19, Lemma 5.5])**.** *$V_{\mathbb{R}}$ is irreducible as a polarizable real variation of Hodge structures.*

Proof. First assume X is smooth. By a result of Matsushita [13, Lemma 2.2] the restriction map $H^2(X, \mathbb{Q}) \to H^2(X_b, \mathbb{Q})$ to a generic fiber of f° is rank one and by Deligne's global invariant cycles theorem $H^2(X, \mathbb{Q}) \to H^0(B^\circ, R^2 f^\circ_* \mathbb{Q}_{X^\circ})$ is surjective [6]. If $V_{\mathbb{R}}$ splits as a variation then the polarizations of the factors would yield a larger than one-dimensional space of sections of $R^2 f^\circ_* \mathbb{R}_{X^\circ} = \wedge^2 V_{\mathbb{R}}$, which is a contradiction.

Now if X is a primitive symplectic variety, one easily checks using the results of [2] that Matsushita's proof carries through verbatim and that $H^2(X, \mathbb{Q}) \to H^0(B^\circ, R^2 f^\circ_* \mathbb{Q}_{X^\circ})$ is still surjective, since the cokernel of $\pi^* : H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$ is generated by exceptional divisors for a log resolution $\pi : Y \to X$ since X has rational singularities. □

Suppose now that f is not of maximal variation. Define the real transcendental lattice $T \subset H^2(X, \mathbb{R})$ to be the polarizable real Hodge substructure spanned by $[\sigma]$ and $[\overline{\sigma}]$. We next claim that the polarizable real variation of Hodge structures $V_{\mathbb{R}} \otimes T^\vee$ has a nontrivial subvariation of level at most one after a finite base-change; the argument below is that of [17], with some mild modifications.

Let $\nu : B^{\circ\prime} \to B^\circ$ be a finite Galois étale cover so that the base-change $V'_{\mathbb{Z}} := \nu^* V_{\mathbb{Z}}$ is pulled back along its period map $\varphi' : B^{\circ\prime} \to S'$, where S' is a level cover of S. Note

[2]Or equivalently, if the monodromy is unitary (by Theorem 2); since there may not be an integral structure, this does not necessarily mean the monodromy is finite.

[3]This definition is equivalent to Beauville's original one.

that up to replacing $B^{\circ\prime}$ with a further finite cover, we may assume φ' can be embedded in a proper map $\overline{\varphi}' : \overline{B}^{\circ\prime} \to S'$ [8]. Denote by $Z \subset S'$ the image of $\overline{\varphi}'$, by $\psi : B^{\circ\prime} \to Z$ the resulting map, and by $V''_{\mathbb{Z}}$ the variation on Z so that $V'_{\mathbb{Z}} = \psi^* V''_{\mathbb{Z}}$. The map $\overline{\varphi}'$ and its image Z are in fact algebraic [4, Theorem 3.1]. We shrink Z (and $B^\circ, B^{\circ\prime}, X^\circ$) so that it is smooth and so that $R^1\psi_*\mathbb{R}_{B^{\circ\prime}}$ is a local system, naturally underlying a graded polarizable real variation of mixed Hodge structures whose only nonzero Hodge components are $(0,0),(1,0),(0,1),(1,1)$ (for example using Saito's theory of mixed Hodge modules [14, 15]). Let $f^{\circ\prime} : X^{\circ\prime} \to B^{\circ\prime}$ be the base-change of f. The natural map $H^2(X,\mathbb{C}) \to H^0(B^{\circ\prime}, R^2 f^{\circ\prime}_*\mathbb{C}_{X^{\circ\prime}})$ sends $[\sigma]$ and hence $T_{\mathbb{C}}$ to zero, since the fibers of f are Lagrangian. By the Leray spectral sequence we have a natural morphism $\mathrm{pt}_Z^* T \to R^1\psi_* V'_{\mathbb{R}} \cong V''_{\mathbb{R}} \otimes R^1\psi_*\mathbb{R}_{B^{\circ\prime}}$ in the category of real variations of mixed Hodge structures. This map is nonzero from the following geometric description as in [17].

Through a very general point $b \in B^{\circ\prime}$, say above a point $z \in Z$, let F be the positive-dimensional fiber of ψ through b. The restricted family X_F is isotrivial, so after replacing F with a finite base-change we can trivialize the monodromy of $V'_{\mathbb{C}}|_F$ and the following natural diagram commutes

$$\begin{tikzcd} T \arrow[r] \arrow[d] & (V''_{\mathbb{C}} \otimes R^1\psi_*\mathbb{C}_{B^{\circ\prime}})_z \arrow[d] \\ H^2(X_F,\mathbb{C}) \arrow[r] & H^1(X_b,\mathbb{C}) \otimes H^1(F,\mathbb{C}) \end{tikzcd}$$

where the bottom arrow comes from the degeneration of the Leray spectral sequence for $X_F \to F$. In the projective case we have $X_F \cong X_b \times F$ (possibly after a further base-change) and this map is just the Künneth projection. The image of $[\sigma]$ is nonzero in the bottom right corner since: (i) σ is nonzero when restricted to X_F since $\dim X_F > \frac{1}{2}\dim X$; (ii) $\sigma|_{X_F}$ extends to a smooth compactification since σ extends to a smooth compactification of X°, so $[\sigma] \neq 0 \in H^2(X_F,\mathbb{C})$; (iii) the image of $[\sigma]$ in $H^2(X_b,\mathbb{C}) \otimes H^0(F,\mathbb{C})$ vanishes and $[\sigma]$ is not in the image of $H^0(X_b,\mathbb{C}) \otimes H^2(F,\mathbb{C})$, as it is not pulled back from F.

Thus, there is a nonzero morphism of real variations

$$V'_{\mathbb{R}} \otimes T^\vee \to \mathrm{gr}^W_{-1}\,\psi^*(R^1\psi_*\mathbb{R}_{B^{\circ\prime}})^\vee. \tag{2}$$

As the category of polarizable real variations of (pure) Hodge structures is semi-simple, we therefore have a splitting

$$V'_{\mathbb{R}} \otimes T^\vee = U \oplus W$$

of real variations, where $U \neq 0$ is the image of (2). In particular, U has level at most one and weight -1.

Now by Lemma 4 the Galois group of ν acts transitively on the isotypic factors of $V'_{\mathbb{R}}$. In particular, if f (and therefore $V_{\mathbb{R}}$) is not isotrivial, no factor of $V'_{\mathbb{R}}$ (as a variation) is isotrivial, or else its entire isotypic component would be, and so would $V_{\mathbb{R}}$. But then there can be no nonzero morphism of variations $V'_{\mathbb{R}} \otimes T^\vee \to U$. Indeed, by Lemma 3, an irreducible factor N of $V'_{\mathbb{C}}$ has level one of degrees 0,1, and $N \otimes T^\vee_{\mathbb{C}}$ can only map nontrivially to an irreducible factor of $U_{\mathbb{C}}$ of the form $N(1)$, while $\mathrm{Hom}(N \otimes T^\vee_{\mathbb{C}}, N(1)) = T_{\mathbb{C}}(1) \cong \mathbb{C}(-1) \oplus \mathbb{C}(1)$ has no degree 0 elements. Thus, f must be isotrivial. $\square$

Remark 5. We revisit the example from [17, §4]. Let $p \geq 5$ be a prime and λ a pth root of unity. Consider a family of abelian varieties $f : X \to B$ with a cyclic automorphism such that the induced automorphism α of $V_{\mathbb{R}} = R^1 f_*\mathbb{R}_X$ has λ as an eigenvalue on $V^{1,0}$ but not on $V^{0,1}$. Let α' be the automorphism of $T^\vee$ with eigenvalue λ^{-1} on $(T^\vee)^{-2,0}$ and eigenvalue λ on $(T^\vee)^{0,-2}$. Then $V_{\mathbb{R}} \otimes T^\vee$ has a level one factor, namely the 1 eigenspace $(V_{\mathbb{R}} \otimes T^{\vee})^1$ of $\alpha \otimes \alpha'$. But the condition on the eigenvalues means the eigenspaces $(V_{\mathbb{C}})^{\lambda}$ and $(V_{\mathbb{C}})^{\lambda^{-1}}$ are level zero, and the real variation $(V_{\mathbb{C}})^{\lambda} \oplus (V_{\mathbb{C}})^{\lambda^{-1}}$ is an isotrivial real factor.

We also obtain the following:

Corollary 6. *Let X be a primitive symplectic variety and $f : X \to B$ a Lagrangian fibration. Let L be a line bundle whose restriction to the smooth fibers is topologically trivial. Then the set of points $b \in B^{\circ}(\mathbb{C})$ for which $L|_{X_b}$ is torsion is analytically dense in B.*

Corollary 6 was proven by Voisin [19, Theorem 1.3] assuming either f is of maximal variation and $\dim X \leq 8$ or isotrivial with no restriction on the dimension. Some applications of Corollary 6 (and more generally Proposition 7 below) to the Chow group and the construction of constant cycle curves are discussed in [19, §1.2].

We deduce this corollary using the following result of Gao. Recall that for a projective family $f : X \to B$ of g-dimensional abelian varieties equipped with a section s and letting $\tilde{B} \to B^{\mathrm{an}}$ be the universal cover, the Betti map $\beta : \tilde{B} \to H_1(X_b, \mathbb{R})$ is the real analytic map obtained by taking the coordinates of the section s with respect to the flat real-analytic trivialization of f. Observe that $\beta^{-1}(H_1(X_b, \mathbb{Q}))$ is the set of points of $\tilde{B}$ at which s is torsion.

Proposition 7 ([7, Theorem 9.1])**.** *Let $f : X \to B$ be a projective family of g-dimensional abelian varieties with $\dim B \geq g$ and $s : B \to X$ a section. Assume f is of maximal variation, that s is non-torsion, and that the very general fiber of f has no nontrivial $\mathbb{Q}$-factor. Then the Betti map $\beta : \tilde{B} \to H_1(X_b, \mathbb{R})$ associated to s is generically submersive.*

Gao proves Proposition 7 as a simple application of the Ax–Schanuel theorem for universal families of abelian varieties [7, Theorem 1.1]. This generalizes the results of André–Corvaja–Zannier [1] which were used in [19].

Proof of Corollary 6. By Voisin's result and Theorem 1 we may assume f is of maximal variation. Consider the family of abelian varieties $h : \mathrm{Pic}^0(X^{\circ}/B^{\circ}) \to B^{\circ}$ and the section $s : b \mapsto L|_{X_b}$. Let $\nu : B^{\circ\prime} \to B^{\circ}$ be a Galois finite base-change for which the $\mathbb{Q}$-factors of the very general fiber of h are defined over $B^{\circ\prime}$. As the Galois group of ν acts transitively on the factors by Lemma 4, the d factors all have the same dimension g', and the image of the period map of each factor must have dimension $\geq g'$, or else the image of the period map of f would have dimension smaller than $dg' = \dim(X^{\circ}/B^{\circ}) = \dim(B^{\circ})$. The base-change of the section s is also Galois invariant, so it suffices to prove the density statement for its projection to a single factor $Y^{\circ} \to B^{\circ\prime}$. Applying Proposition 7, the Betti map $\beta : \tilde{B} \to H_1(Y_b, \mathbb{R})$ is submersive, so $\beta^{-1}(H_1(Y_b, \mathbb{Q}))$ is analytically dense in $\tilde{B}$ as claimed. □

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