LAGRANGIAN 4-PLANES IN HOLOMORPHIC SYMPLECTIC VARIETIES OF $K3^{[4]}$-TYPE

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Abstract. We classify the cohomology classes of Lagrangian 4-planes $\mathbb{P}^4$ in a smooth manifold X deformation equivalent to a Hilbert scheme of 4 points on a $K3$ surface, up to the monodromy action. Classically, the Mori cone of effective curves on a $K3$ surface S is generated by nonnegative classes C, for which $(C,C)\geq 0$, and nodal classes C, for which $(C,C)=-2$; Hassett and Tschinkel conjecture that the Mori cone of a holomorphic symplectic variety X is similarly controlled by "nodal" classes C such that $(C,C)=-\gamma$, for $(\cdot,\cdot)$ now the Beauville–Bogomolov form, where γ classifies the geometry of the extremal contraction associated to C. In particular, they conjecture that for X deformation equivalent to a Hilbert scheme of n points on a $K3$ surface, the class $C=\ell$ of a line in a smooth Lagrangian n-plane $\mathbb{P}^n$ must satisfy $(\ell,\ell)=-\frac{n+3}{2}$. We prove the conjecture for $n=4$ by computing the ring of monodromy invariants on X, and showing there is a unique monodromy orbit of Lagrangian 4-planes.

1. Introduction

Let X be an irreducible holomorphic symplectic variety; thus, X is a smooth projective simply-connected variety whose space $H^0(\Omega^2_X)$ of global two-forms is generated by a nowhere degenerate form ω. $H^2(X,\mathbb{Z})$ carries a deformation-invariant nondegenerate primitive integral form $(\cdot,\cdot)$ called the Beauville–Bogomolov form [Bea83]. For $X=S$ a $K3$ surface $(\cdot,\cdot)$ is the intersection form, while for $X=S^{[n]}$ a Hilbert scheme of $n>1$ points on S we have the orthogonal decomposition [Bea83, §8]

$$H^2(S^{[n]},\mathbb{Z})_{(\cdot,\cdot)}\cong H^2(S,\mathbb{Z})\oplus_\perp \mathbb{Z}\delta \tag{1}$$

where the form on $H^2(S,\mathbb{Z})$ is the intersection form, 2δ is the divisor of non-reduced subschemes, and $(\delta,\delta)=2-2n$. The embedding of $H^2(S,\mathbb{Z})$ is achieved via the canonical isomorphism

$$H^2(S,\mathbb{Z})\cong H^2(\mathrm{Sym}^n S,\mathbb{Z})$$

and pullback along the contraction $\sigma: S^{[n]}\to \mathrm{Sym}^n S$. The inverse of $(\cdot,\cdot)$ defines a $\mathbb{Q}$-valued form on $H_2(X,\mathbb{Z})$ which we will also denote $(\cdot,\cdot)$; by Poincaré duality, we obtain a decomposition dual to (1). For example, the class $\delta^\vee\in H_2(X,\mathbb{Z})$ Poincaré dual to the exceptional divisor δ has square $(\delta^\vee,\delta^\vee)=\frac{1}{2-2n}$. The form induces an embedding $H^2(X,\mathbb{Z})\subset H_2(X,\mathbb{Z})$ under which the two forms match up, and since the determinant of $(\cdot,\cdot)$ on $H^2(X,\mathbb{Z})$ is $2-2n$, we can write any $\ell\in H_2(X,\mathbb{Z})$ as $\ell=\frac{\lambda}{2n-2}$ for some $\lambda\in H^2(X,\mathbb{Z})$. We will refer to the smallest multiple of ℓ that is in $H^2(X,\mathbb{Z})$ as the Beauville–Bogomolov dual ρ of ℓ.

1.1. **Cones of effective curves.** Much of the geometry of a $K3$ surface S is encoded in its nodal classes, the indecomposable effective curve classes C for which $(C,C)=-2$. Suppose S has an ample divisor H; let $N_1(S,\mathbb{Z})\subset H_2(S,\mathbb{Z})$ be the group of curve

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classes modulo homological equivalence, and $\mathrm{NE}_1(S) \subset N_1(S,\mathbb{R}) = N_1(S,\mathbb{Z}) \otimes \mathbb{R}$ the Mori cone of effective curves. It is well-know that [LP80, Lemma 1.6]

$$\mathrm{NE}_1(S) = \langle C \in N_1(S,\mathbb{Z}) | H \cdot C > 0 \text{ and } C \cdot C \geq -2 \rangle \tag{2}$$

By Kleiman's criterion there is a dual statement for the ample cone; here by $\langle \cdots \rangle$ we mean "the cone generated by $\cdots$".

Hassett and Tschinkel [HT10b] conjectured that the cone of effective curves in an irreducible holomorphic symplectic variety X is similarly determined intersection theoretically by the Beauville–Bogomolov form. The original form of the conjecture was incorrect[1], though it has been proven in spirit due to work of Bayer–Macrì [BM13] (for moduli spaces) and Bayer–Hassett–Tschinkel [BHT13] and Mongardi [Mon13] (for general irreducible holomorphic symplectic manifolds). In particular,

Theorem 1.2. *(*cf. [BHT13, Proposition 2]*) Let X be deformation-equivalent to the Hilbert scheme of n points on a K3 surface (*i.e. *of $K3^{[n]}$-type) with polarization H. Then each extremal ray of the Mori cone $\mathrm{NE}_1(X)$ contains an effective curve class R such that*

$$(R,R) \geq -\frac{n+3}{2}$$

Remark 1.3. An exact formula for the Mori cone in terms of Markman's extended weight two Hodge structure is given by [BM13, Theorem 12.2] and [BHT13, Theorem 1]. We don't state the full theorem here mainly to avoid the notational tangent.

Hassett and Tschinkel further conjecture that much of the geometry of the "nodal" classes—extremal classes of negative Beauville–Bogomolov square—is determined by their intersection theory. The case when X is deformation equivalent to a Hilbert scheme of 2 points on a $K3$ surface is worked out in full detail in [HT09, Theorem 1]: the three types of indecomposable "nodal" classes have Beauville–Bogomolov square $-\frac{1}{2}, -2$, and $-\frac{5}{2}$ and their extremal rays correspond to the 2 types of extremal contractions:

(i) Divisorial extremal contractions. In this case, the exceptional locus is a divisor E is contracted to a $K3$ surface T. The generic fiber over T is either an A_1 or A_2 configuration of rational curves [HT09, Theorem 21], and if C is the class of the generic fiber of the normalization, then either $(C,C) = -2$ or $-1/2$, respectively.

(ii) Small extremal contractions. In this case, the exceptional locus is a Lagrangian $\mathbb{P}^2$ contracted to an isolated singularity, and the class of a line ℓ satisfies $(\ell,\ell) = -5/2$.

See [HT10b] for some speculations about the "nodal" classes that appear in higher dimensions.

1.4. **Lagrangian n-planes.** Generalizing slightly, let X be an irreducible holomorphic symplectic *manifold*—that is, a simply-connected Kähler manifold with $H^0(\Omega^2_X) \cong \mathbb{C}$ generated by a nowhere degenerate 2-form. There are only two infinite families of deformation classes of irreducible holomorphic symplectic manifolds known: Hilbert schemes of points on $K3$ surfaces and generalized Kummer varieties. The first piece of the Hassett–Tschinkel program is a generalization of (ii) above.

Conjecture 1. *(*[HT10b, Conjecture 1.2]*) Let X be of $K3^{[n]}$-type, $\mathbb{P}^n \subset X$ a smoothly embedded Lagrangian n-plane, and $\ell \in H_2(X,\mathbb{Z})$ the class of the line in $\mathbb{P}^n$. Then*

$$(\ell,\ell) = -\frac{n+3}{2}$$

[1] A counterexample was originally constructed by Markman [Mar] for X deformation equivalent to a Hilbert scheme of 5 points on a $K3$ surface, and the example is treated in detail in [BM12, Remark 9.4]. For X of dimension < 8 the original form of the conjecture is still expected to be true.

In view of Theorem 1.2, we can view these curve classes as the "most extremal." The conjecture has been verified for $n = 2$ in [HT09] and for $n = 3$ in [HHT].

Remark 1.5. There is a similar conjecture for the class of a line ℓ in a smoothly embedded Lagrangian n-plane $\mathbb{P}^n \subset X$ for X deformation equivalent to a $2n$-dimensional generalized Kummer variety K_nA of an abelian surface A. In this case, we expect

$$(\ell, \ell) = -\frac{n+1}{2}$$

This conjecture has been verified for $n = 2$ in [HT10a].

Our main result is a proof of Conjecture 1 in the $n = 4$ case; furthermore, we determine the class of the Lagrangian 4-plane:

Theorem 1.6 (see Theorem 4.4)**.** *Let X be of $K3^{[4]}$-type, $\mathbb{P}^4 \subset X$ be a smoothly embedded Lagrangian 4-plane, $\ell \in H_2(X, \mathbb{Z})$ the class of a line in $\mathbb{P}^4$, and $\rho = 2\ell \in H^2(X, \mathbb{Q})$. Then ρ is integral, and*

$$[\mathbb{P}^4] = \frac{1}{337920}\left(880\rho^4 + 1760\rho^2 \mathrm{c}_2(X) - 3520\theta^2 + 4928\theta \mathrm{c}_2(X) - 1408\mathrm{c}_2(X)^2\right)$$

Further, we must have $(\ell, \ell) = -\frac{7}{2}$.

Here θ is the image of the dual to the Beauville–Bogomolov form, thought of as an element of $\mathrm{Sym}^2 H_2(X, \mathbb{Q})^* \cong \mathrm{Sym}^2 H^2(X, \mathbb{Q})$, under the cup product map $\mathrm{Sym}^2 H^2(X, \mathbb{Q}) \to H^4(X, \mathbb{Q})$. Likewise in the $n = 3$ case the class of the Lagrangian 3-plane is completely determined by ℓ, *cf.* [HHT, Theorem 1.1]. Our theorem provides evidence that Conjecture 1 is true in general, and conjecturally determines the minimal Beauville–Bogomolov square of indecomposable nodal classes on eightfolds deformation equivalent to Hilbert schemes of points on $K3$ surfaces.

1.7. **Monodromy.** We prove our result by using the representation theory of the monodromy group of X to relate the intersection theory of X to that of a Hilbert scheme of 4 points on a $K3$ surface, where the cohomology ring is actually computable. In doing so we completely determine the ring of monodromy invariants on X.

Recall that a monodromy operator is the parallel translation operator on $H^*(X, \mathbb{Z})$ associated to a smooth family of deformations of X; the monodromy group $\mathrm{Mon}(X)$ is the subgroup of $\mathrm{GL}(H^*(X, \mathbb{Z}))$ generated by all monodromy operators. Let $\mathrm{Mon}^2(X) \subset \mathrm{GL}(H^2(X, \mathbb{Z}))$ be the quotient acting nontrivially on degree 2 cohomology, and $\overline{\mathrm{Mon}}(X) \subset \mathrm{GL}(H^*(X, \mathbb{C}))$ (respectively $\overline{\mathrm{Mon}^2}(X) \subset \mathrm{GL}(H^2(X, \mathbb{C}))$) the Zariski closure of $\mathrm{Mon}(X)$ (respectively $\mathrm{Mon}^2(X)$). By the deformation invariance of the Beauville–Bogomolov form, $\mathrm{Mon}^2(X)$ is actually contained in $\mathrm{O}(H^2(X, \mathbb{Z}))$, the orthogonal group of $H^2(X, \mathbb{Z})$ with respect to $(\cdot, \cdot)$. *A priori*, the full Lie group $G_X = \mathrm{SO}(H^2(X, \mathbb{C}))$ only acts on $H^2(X, \mathbb{C})$, but in fact for X of $K3^{[n]}$-type, the full cohomology ring $H^*(X, \mathbb{C})$ carries a representation of $G_X = \mathrm{SO}(H^2(X, \mathbb{C}))$ compatible with cup product ([HHT, Proposition 4.1]). The basic reason for this is two-fold, both results of Markman:

(a) the quotient $\mathrm{Mon}(X) \to \mathrm{Mon}^2(X)$ has finite kernel [Mar08, §4.3];
(b) G_X is a connected component of $\overline{\mathrm{Mon}^2}(X)$ [Mar08, §1.8].

The representation of $\mathrm{Mon}(X)$ on $H^*(X, \mathbb{C})$ extends to one of $\overline{\mathrm{Mon}}(X)$. By the above the connected component of the universal covers of $\overline{\mathrm{Mon}}(X), \overline{\mathrm{Mon}^2}(X)$ and G_X are all identified, so the universal cover of G_X acts on all of $H^*(X, \mathbb{C})$; the representation descends to G_X because of the vanishing of odd cohomology.

The action respects the Hodge structure, so we may consider the ring of Hodge classes:

$$I^*(X) = H^*(X, \mathbb{Q}) \cap H^*(X, \mathbb{C})^{G_X}$$

Of course, $I^*(X)$ contains the Chern classes of the tangent bundle of X and the Beauville–Bogomolov class $\theta \in H^4(X,\mathbb{Q})$, but there can be many other Hodge classes. Markman [Mar11] constructs another series of Hodge classes $k_i \in I^{2i}(X)$, $i \geq 2$, as characteristic classes of monodromy-invariant twisted sheaves.

Given $\lambda \in H^2(X,\mathbb{Q})$, let $G_\lambda \subset G_X$ be the stabilizer of λ. Define

$$I^*_\lambda(X) = H^*(X,\mathbb{Q}) \cap H^*(X,\mathbb{C})^{G_\lambda}$$

to be the ring of cohomology classes invariant under the monodromy group preserving λ. For example, given a Lagrangian n-plane $\mathbb{P}^n \subset X$, the deformations of X that deform $\mathbb{P}^n$ are precisely those in $H^{1,1}(X) \cap \rho^\perp$, where ρ is the Beauville–Bogomolov dual of the class of the line in $\mathbb{P}^n$, and the orthogonal is taken with respect to the Beauville–Bogomolov form [Ran95, Voi92]. Thus, the class $[\mathbb{P}^n] \in H^{2n}(X,\mathbb{Z})$ must lie in the subring $I^*_\rho(X)$. G_X will act on these cohomology classes, and up to this action we expect there is a unique Lagrangian n-plane in general. For $n = 4$, this is a consequence of our result since G_X acts transitively on rays in $H^2(X,\mathbb{C})$:

Corollary 1.8. *For X of $K3^{[4]}$-type, there is a unique G_X orbit of smooth Lagrangian 4-plane classes $[\mathbb{P}^4] \in H^8(X,\mathbb{C})$.*

Method of Proof and Outline. We prove our result by first completely determining $I^*_\lambda(X)$ for $X = S^{[4]}$ a Hilbert scheme of 4 points on a $K3$ surface S and $\lambda = \delta$. This is done in Section 1 using the Nakajima basis and the results of [LS03] on cup product. The ring $I^*_\lambda(X)$ in the general case of X of $K3^{[4]}$-type and $\lambda \in H^2(X,\mathbb{Z})$ will be isomorphic since G_X acts transitively on rays in $H^2(X,\mathbb{Z})$. In Section 2 we construct an explicit isomorphism by finding a monodromy invariant basis for $I^*_\lambda(X)$, from which we are able to derive the intersection form on $I^8_\lambda(X)$. In Section 3 we take λ proportional to the Beauville–Bogomolov dual of the class of a line in a smooth Lagrangian 4-plane $\mathbb{P}^4 \subset X$ and produce a diophantine equation in the coefficients of the class $[\mathbb{P}^4]$ with respect to the basis from Section 2. In Section 4, we show the only solution to the diophantine equation is the conjectural one. For completeness we include an appendix summarizing our localization computations to calculate the Fujiki constants in Section 2.

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2. Structure of the ring of monodromy invariants

2.1. **The Lehn-Sorger formalism.** We briefly summarize the work of Lehn and Sorger in [LS03] on the cohomology ring of a Hilbert scheme of points on a $K3$ surface. Given a Frobenius algebra A, they construct a Frobenius algebra $A^{[n]}$ such that when $A = H^*(S,\mathbb{Q})$ for S a $K3$ surface, $A^{[n]}$ is canonically $H^*(S^{[n]},\mathbb{Q})$.

The algebra $A = H^*(S,\mathbb{Q})$ comes equipped with a form $T = -\int_S : A \to \mathbb{Q}$ and a multiplication $m : A \otimes A \to A$ (given by cup-product) such that the pairing $(x,y) = T(xy)$ is nondegenerate. There is also a comultiplication $\Delta : A \to A \otimes A$ adjoint to m with respect to the form $T \otimes T$ on $A \otimes A$. In this case Δ is the push-forward along the diagonal. Writing $1 \in H^0(S,\mathbb{Z})$ for the unit, $[\mathrm{pt}] \in H^4(S,\mathbb{Z})$ for the point class, $e_1, \ldots, e_{22}$ as a basis for $H^2(S,\mathbb{Z})$, and $e_1^\vee, \ldots, e_{22}^\vee$ for the dual basis with respect to the intersection form, a simple computation using adjointness shows that $\Delta(1) = -\sum_j e_j \otimes e_j^\vee - [\mathrm{pt}] \otimes 1 - 1 \otimes [\mathrm{pt}]$, $\Delta(e_j) = -e_j \otimes [\mathrm{pt}] - [\mathrm{pt}] \otimes e_j$, $\Delta(e_j^\vee) = -e_j^\vee \otimes [\mathrm{pt}] - [\mathrm{pt}] \otimes e_j^\vee$ and $\Delta([\mathrm{pt}]) = -[\mathrm{pt}] \otimes [\mathrm{pt}]$. Thus $\mathbf{e} = -24[\mathrm{pt}]$.

We also have an n-fold multiplication $m[n] : A^{\otimes n} \to A$ and its adjoint $\Delta[n] : A \to A^{\otimes n}$. Note that $m[1] = \Delta[1] = \mathrm{id}$, $m[2] = m$, and $\Delta[2] = \Delta$.

Lemma 2.2. *Using the previous formulae one obtains:*

$$\begin{aligned}
\Delta[3](1) &= \sum_j \sum (e_j)_a \otimes (e_j^\vee)_b \otimes [\mathrm{pt}]_c + \sum [\mathrm{pt}]_a \otimes [\mathrm{pt}]_b \otimes 1_c \\
\Delta[3](e_j) &= \sum [\mathrm{pt}]_a \otimes [\mathrm{pt}]_b \otimes (e_j)_c \\
\Delta[3](e_j^\vee) &= \sum [\mathrm{pt}]_a \otimes [\mathrm{pt}]_b \otimes (e_j^\vee)_c \\
\Delta[3]([\mathrm{pt}]) &= [\mathrm{pt}] \otimes [\mathrm{pt}] \otimes [\mathrm{pt}]
\end{aligned}$$

By $[\mathrm{pt}]_a \otimes [\mathrm{pt}]_b \otimes 1_c \in A^{\otimes 3}$ we mean $[\mathrm{pt}]$ inserted in the ath and bth tensor factors, and 1 inserted in the cth factor. All unspecified sums in Lemma 2.2 are over bijections $\{1,2,3\} \xrightarrow{\cong} \{a,b,c\}$.

Proof. This follows from the relation $m[n] = m[2] \circ (m[n-1] \otimes \mathrm{id})$ for $n \geq 2$ and the dual relation $\Delta[n] = (\Delta[n-1] \otimes \mathrm{id}) \circ \Delta[2]$. □

Let $[n] = \{k \in \mathbb{N} | k \leq n\}$. Define the tensor product of A indexed by a finite set I of cardinality n as

$$A^I := \left(\bigoplus_{\varphi : [n] \xrightarrow{\cong} I} A_{\varphi(1)} \otimes \cdots \otimes A_{\varphi(n)} \right) / S_n$$

where S_n acts by permuting the tensor factors in each summand in the obvious way. A^I is a Frobenius algebra with multiplication m^I and form T^I.

Note that for (finite) sets U, V and a bijection $U \to V$ there is a canonical isomorphism $A^U \to A^V$, so we can always choose a bijection of I with some $[k]$ to reduce to the usual notion of finite self tensor products. In general, for any surjection $\varphi : U \to V$, there is an obvious ring homomorphism

$$\varphi^* : A^U \to A^V$$

using the ring structure to combine factors indexed by elements of U in the same fiber of φ. There is an adjoint map

$$\varphi_* : A^V \to A^U$$

with the important relation

$$\varphi_*(a \cdot \varphi^*(b)) = \varphi_*(a) \cdot b$$

which follows directly from the adjointness.

For any subgroup $G \subset S_n$, we can consider the left coset space $G \backslash [n]$, and form $A^{G\backslash[n]}$. In particular, for $\sigma \in S_n$ and $G = \langle \sigma \rangle$ the group generated by σ, we denote $A^\sigma = A^{G\backslash[n]}$. Let

$$A\{S_n\} = \bigoplus_{\sigma \in S_n} A^\sigma \cdot \sigma$$

A pure tensor element of A^σ is specified by attaching an element $\alpha_i \in A$ to each orbit $i \in I = \langle \sigma \rangle \backslash [n]$. For example, for a function $\nu : I \to \mathbb{Z}_{\geq 0}$,

$$\mathbf{e}^\nu = \otimes_{i \in I} \mathbf{e}^{\nu(i)} \in A^\sigma$$

There is a natural product structure on $A\{S_n\}$. For any inclusion of subgroups $H \subset K$ of S_n there is a surjection $H\backslash[n] \to K\backslash[n]$ and therefore maps

$$f^{H,K} : A^{H\backslash[n]} \to A^{K\backslash[n]}$$

$$f_{K,H} : A^{K\backslash[n]} \to A^{H\backslash[n]}$$

The product is then

$$\begin{aligned} A^\sigma \otimes A^\tau &\longrightarrow && A^{\sigma\tau} \\ a \otimes b &\longrightarrow && f_{\langle\sigma,\tau\rangle,\langle\sigma\tau\rangle}\left(f^{\langle\sigma\rangle,\langle\sigma,\tau\rangle}(a) \cdot f^{\langle\tau\rangle,\langle\sigma,\tau\rangle}(b) \cdot \mathbf{e}^{g(\sigma,\tau)}\right) \end{aligned} \tag{3}$$

where $\langle\sigma,\tau\rangle$ is the subgroup of S_n generated by σ,τ, and the graph defect $g(\sigma,\tau) : \langle\sigma,\tau\rangle\backslash[n] \to \mathbb{Z}_{\geq 0}$ is

$$g(\sigma,\tau)(B) = \frac{1}{2}\left(|B| + 2 - |\langle\sigma\rangle\backslash B| - |\langle\tau\rangle\backslash B| - |\langle\sigma\tau\rangle\backslash B|\right)$$

S_n acts naturally on $A\{S_n\}$. For any $\tau \in A\{S_n\}$, there is for any $\sigma \in S_n$ a bijection $\tau : \langle\sigma\rangle\backslash[n] \to \langle\tau\sigma\tau^{-1}\rangle\backslash[n]$. τ then acts on $A\{S_n\}$ via $\tau^* : A^\sigma \cdot \sigma \to A^{\tau\sigma\tau^{-1}} \cdot \tau\sigma\tau^{-1}$ on each factor. Define

$$A^{[n]} = A\{S_n\}^{S_n}$$

Note that for any partition $\mu = (1^{\mu_1}, 2^{\mu_2}, \cdots)$ of n, there is a piece

$$A^{[n]}_\mu = \left(\bigoplus_{\sigma\in C_\mu} A^\sigma \cdot \sigma\right)^{S_n} \cong \bigotimes_i \operatorname{Sym}^{\mu_i} A \tag{4}$$

where $C_\mu \subset S_n$ is the conjugacy class of permutations σ of cycle type μ.

If A is a graded Frobenius algebra, then $A^{[n]}$ is naturally graded. A^σ is graded as a tensor product of graded vector spaces, and we take

$$A^\sigma \cdot \sigma \cong A^\sigma[-2|\sigma|]$$

where if the cycle type of σ is μ, $|\sigma| = \sum_i (i-1)\mu_i$. In particular, the mth graded piece of (4) is

$$(A^{[n]}_\mu)_m \cong \bigoplus_{\substack{(w,\mu) \\ |(w,\mu)|=m}} \bigotimes_i \operatorname{Sym}^{\mu_i} A_{w_i} \tag{5}$$

where the sum is taken over weighted permutations (w,μ)—*i.e.* a partition μ and a weight w_i associated to each part—with

$$m = |(w,\mu)| = \sum_i (i-1)\mu_i + w_i$$

We then have

Theorem 2.3. *(*[LS03, Theorem 1.1]*) For S a $K3$ surface, there is a natural isomorphism of graded Frobenius algebras*

$$(H^*(S,\mathbb{Q})[2])^{[n]} \cong H^*(S^{[n]},\mathbb{Q})[2n]$$

The grading shift on both sides is such that the 0th graded piece is middle cohomology.

Remark 2.4. It will be important in the next section to note that under the isomorphism of Theorem 2.3,

$$n![\mathrm{pt}]_1 \otimes \cdots \otimes [\mathrm{pt}]_n \cdot (\mathrm{id}) \mapsto [\mathrm{pt}]_{S^{[n]}} \tag{6}$$

2.5. **Monodromy invariants.** Let S be a $K3$ surface, and $G_S = \mathrm{SO}(H^2(S,\mathbb{C}))$ the special orthogonal group of the intersection form $(\cdot,\cdot)$ on S. $H^*(S,\mathbb{C})$ is naturally a representation of G_S, acting via the standard representation on $H^2(S,\mathbb{C})$ and the trivial representations on $H^0(S,\mathbb{C})$ and $H^4(S,\mathbb{C})$.

Recall (see for example [FH91]) that positive weights of the algebra $\mathrm{SO}_{\mathbb{C}}(k)$ of rank r ($k = 2r$ or $2k+1$) are r-tuples $\lambda = (\lambda_1, \dots, \lambda_r)$ with the λ_i either all integral or all half-integral, and either

$$\begin{aligned} \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r-1} \geq |\lambda_r| \geq 0, \qquad & k = 2r \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r-1} \geq \lambda_r \geq 0, \qquad & k = 2r+1 \end{aligned}$$

Let the representation of $\mathrm{SO}_{\mathbb{C}}$ of highest weight λ be denoted $V(\lambda)$. Thus, $\mathbf{1} = V(0,\dots)$ is the trivial representation, and $V = V(1,0,\dots)$ the standard. $\mathrm{Sym}^k V$ is not irreducible, since the form yields an invariant $\theta \in \mathrm{Sym}^2 V$, but $V(k,0,\dots) = \mathrm{Sym}^k V / \mathrm{Sym}^{k-2} V$. In the sequel, we will only indicate the nonzero weights, *e.g.* $V = V(1)$.

If a Frobenius algebra A carries a representation of a group G, $A^{[n]}$ naturally carries a representation of G that can easily be read off of (5). Thus,

Proposition 2.6. *As a representation of G_S, we have*

$$\begin{aligned} H^2(S^{[4]},\mathbb{C}) &\cong \mathbf{1}_S \oplus V_S(1) \\ H^4(S^{[4]},\mathbb{C}) &\cong \mathbf{1}_S^4 \oplus V_S(1)^2 \oplus V_S(2) \\ H^6(S^{[4]},\mathbb{C}) &\cong \mathbf{1}_S^5 \oplus V_S(1)^5 \oplus V_S(1,1) \oplus V_S(2)^2 \oplus V_S(3) \\ H^8(S^{[4]},\mathbb{C}) &\cong \mathbf{1}_S^8 \oplus V_S(1)^6 \oplus V_S(1,1) \oplus V_S(2)^4 \oplus V_S(2,1) \oplus V_S(3) \oplus V_S(4) \end{aligned}$$

Poincaré duality is compatible with the G_S action, so the above determines all cohomology groups.

Note that the invariant class in $H^2(S^{[n]},\mathbb{C})$ is exactly δ. The decomposition (1) identifies the action of G_S on $H^*(S^{[n]},\mathbb{C})$ with that of $G_\delta \subset G_{S^{[n]}}$, the stabilizer of δ. In other words, deformations of $S^{[n]}$ orthogonal to the exceptional divisor δ remain Hilbert schemes of points of a $K3$ surface, and therefore come from a deformation of S.

Recall that $\mathrm{SO}_{\mathbb{C}}(k)$ has universal branching rules. For $\mathrm{SO}_{\mathbb{C}}(k-1) \subset \mathrm{SO}_{\mathbb{C}}(k)$ the stabilizer of a nonisotropic vector $v \in V$, $(v,v) \neq 0$, we have

$$\mathrm{Res}^{\mathrm{SO}_{\mathbb{C}}(k)}_{\mathrm{SO}_{\mathbb{C}}(k-1)} V(\lambda) = \bigoplus_{\lambda'} V(\lambda')$$

where the sum is taken over all weights λ' with

$$\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \geq \cdots \geq \lambda_r \geq |\lambda'_r| \geq 0$$

For X of $K3^{[n]}$-type, we can therefore deduce the structure of $H^*(X,\mathbb{C})$ as a G_X representation from the structure of $H^*(S^{[n]},\mathbb{C})$ as a G_S representation:

Corollary 2.7. *For X of $K3^{[4]}$-type,*

$$\begin{aligned} H^2(X,\mathbb{C}) &\cong V_X(1) \\ H^4(X,\mathbb{C}) &\cong \mathbf{1}_X^2 \oplus V_X(1) \oplus V_X(2) \\ H^6(X,\mathbb{C}) &\cong \mathbf{1}_X \oplus V_X(1)^2 \oplus V_X(1,1) \oplus V_X(2) \oplus V_X(3) \\ H^8(X,\mathbb{C}) &\cong \mathbf{1}_X^3 \oplus V_X(1)^2 \oplus V_X(2)^2 \oplus V_X(2,1) \oplus V_X(4) \end{aligned}$$

Again, Poincaré duality determines the representations of the other cohomology groups.

2.8. **A basis for $I^*_\delta(S^{[4]})$.** For a partition $\mu = (1^{\mu_1}, 2^{\mu_2}, \ldots)$ of n, the number of parts of μ is $\ell(\mu) = \sum \mu_i$. By a labelled partition μ we will mean a partition μ and an ordered list of $\ell(\mu)$ cohomology classes $\alpha \in H^*(S, \mathbb{Q})$. For example, $(\{1\}_2, \{1,1\}_1)$ is a labelled partition of 4, subordinate to the partition $\mu = (1^2, 2)$, and attaching the unit class to each part of μ. Such a labelled partition μ determines an element of the Lehn-Sorger algebra of $H^*(S,\mathbb{Q})[2]$ by summing over all group elements $\sigma \in S_n$ with cycle type μ, for example

$$\begin{aligned} I(\{1\}_2, \{1,1\}_1) &= \sum_{(12)} 1_{12} \otimes 1_3 \otimes 1_4 (12) \\ &= 1_{12} \otimes 1_3 \otimes 1_4(12) + 1_{13} \otimes 1_2 \otimes 1_4(13) + 1_{14} \otimes 1_2 \otimes 1_3(14) \\ &\quad + 1_1 \otimes 1_{23} \otimes 1_4(23) + 1_1 \otimes 1_{24} \otimes 1_3(24) + 1_1 \otimes 1_2 \otimes 1_{34}(34) \end{aligned}$$

We can generate homogeneous classes of $H^*(S^{[n]}, \mathbb{Q})$ invariant under G_S from partitions of n labelled by cohomology classes $\{1, e, e^\vee, [\mathrm{pt}]\}$, where every time we have a label e, there must be a paired $e^\vee$ label, corresponding to inserting e_j and $e_j^\vee$ in the corresponding tensor factors and summing over j. For example, $I^2_\delta(S^{[4]})$ is spanned by $\delta = I(\{1\}_2, \{1,1\}_1)$. Generating sets for $I^{2k}_\delta(S^{[4]})$ for $k = 2,3,4$ are given by:

$I^4_\delta(S^{[4]})$	$I^6_\delta(S^{[4]})$	$I^8_\delta(S^{[4]})$
$W = I(\{1\}_3, \{1\}_1)$	$P = I(\{1\}_4)$	$A = I(\{e\}_3, \{e^\vee\}_1)$
$X = I(\{1,1\}_2)$	$Q = I(\{[\mathrm{pt}]\}_2, \{1,1\}_1)$	$B = I(\{1\}_3, \{[\mathrm{pt}]\}_1)$
$Y = I(\{1,1,1,[\mathrm{pt}]\}_1)$	$R = I(\{1\}_2, \{1,[\mathrm{pt}]\}_1)$	$C = I(\{[\mathrm{pt}]\}_3, \{1\}_1)$
$Z = I(\{1,1,e,e^\vee\}_1)$	$S = I(\{e^\vee\}_2, \{e,1\}_1)$	$D = I(\{1,[\mathrm{pt}]\}_2)$
	$T = I(\{1\}_2, \{e,e^\vee\}_1)$	$E = I(\{e,e^\vee\}_2)$
		$F = I(\{1,1,[\mathrm{pt}],[\mathrm{pt}]\}_1)$
		$G = I(\{1,e,e^\vee,[\mathrm{pt}]\}_1)$
		$H = I(\{e,e,e^\vee,e^\vee\}_1)$

These classes are all clearly independent, and therefore by the computation of the dimensions of $I^*_\delta(S^{[4]})$ in the previous section they are bases.

2.9. **Cup product on $I^*_\delta(S^{[4]})$.** Using (3) we compute the multiplicative structure of $I^*_\delta(S^{[4]})$ in the above basis. These computations are straightforward; for example,

$$\begin{aligned} \delta^2 &= \left(\sum_{(12)} 1_{12} \otimes 1_3 \otimes 1_4(12) \right)^2 \\ &= \sum_{(12)} \Big(\Delta(1)_{1,2} \otimes 1_3 \otimes 1_4(\mathrm{id}) + 1_{1,2,3} \otimes 1_4(132) \\ &\qquad + 1_{1,2,4} \otimes 1_3(142) + 1_{1,2,3} \otimes 1_4(123) + 1_{1,2,4} \otimes 1_3(124) + 1_{12} \otimes 1_{34}(12)(34) \Big) \\ &= -3 \sum_1 [\mathrm{pt}]_1 \otimes 1_2 \otimes 1_3 \otimes 1_4(\mathrm{id}) - \sum_{(12)} \sum_j (e_j)_1 \otimes (e_j^\vee)_2 \otimes 1_3 \otimes 1_4(\mathrm{id}) \\ &\qquad + 3 \sum_{(123)} 1_{123} \otimes 1_4(123) + 2 \sum_{(12)(34)} 1_{12} \otimes 1_{34}(12)(34) \\ &= -3Y - Z + 3W + 2X \end{aligned}$$

The multiplication table for degree 4 elements is:

	W	X	Y	Z
W	$-3A-3B-27C-8D$ $-8E+4F+2G$	$-3A-3B-3C$	$B+3C$	$3A+66C$
X		$-2D-2E+2F+G+H$	$2D$	$22D+4E$
Y			$2F$	G
Z				$22F+2G+2H$

In particular, note that:

$$\delta^4 = (\delta^2)^2 = -81A - 81B - 729C - 192D - 96E + 84F + 30G + 6H \tag{7}$$

The multiplication table for A, B, C, D, E, F, G, H is much simpler,

	A	B	C	D	E	F	G	H
A	$\frac{176}{24}$	0	0	0	0	0	0	0
B		0	$\frac{8}{24}$	0	0	0	0	0
C			0	0	0	0	0	0
D				$\frac{6}{24}$	0	0	0	0
E					$\frac{66}{24}$	0	0	0
F						$\frac{6}{24}$	0	0
G							$\frac{264}{24}$	0
H								$\frac{1584}{24}$

where we have identified top cohomology $H^{16}(S^{[4]}, \mathbb{Q}) \cong \mathbb{Q}$ as usual via the point class $[\mathrm{pt}]_{S^{[4]}} = 24[\mathrm{pt}]_1 \otimes [\mathrm{pt}]_2 \otimes [\mathrm{pt}]_3 \otimes [\mathrm{pt}]_4(\mathrm{id})$ from (6). As a consistency check, from Corollary 3.3 we have $\delta^8 = 105(\delta, \delta)^4 = 136080$ and indeed, from (7), $\delta^8 = (-81A - 81B - 729C - 192D - 96E + 84F + 30G + 6H)^2 = 136080$. Note that the remaining classes and products (of cohomological degree divisible by 4, which is all we need) are determined by Poincaré duality.

2.10. **The Beauville–Bogomolov form.** From (1), we can explicitly write down θ in the W, X, Y, Z basis:

$$\begin{aligned}\theta &= \sum_j \left(\sum_1 (e_j)_1 \otimes 1_2 \otimes 1_3 \otimes 1_4(\mathrm{id}) \right) \cdot \left(\sum_1 (e_j^\vee)_1 \otimes 1_2 \otimes 1_3 \otimes 1_4(\mathrm{id}) \right) - \frac{1}{6}\delta^2 \\ &= -\frac{1}{2}W - \frac{1}{3}X + \frac{45}{2}Y + \frac{13}{6}Z \end{aligned} \tag{8}$$

By direct compoutation, using the results of the previous section,

Lemma 2.11.

$$\begin{aligned}\theta^4 &= 450225 \\ \delta^2\theta^3 &= -117450 = 19575(-6) \\ \delta^4\theta^2 &= 84564 = 2349 \cdot (-6)^2 \\ \delta^6\theta &= -93960 = 435 \cdot (-6)^3 \\ \delta^8 &= 136080 = 105 \cdot (-6)^4\end{aligned}$$

3. Hodge classes on X

Let X be of $K3^{[4]}$-type and $\lambda \in H^2(X, \mathbb{Q})$. The rings $I^*(X)$ and $I^*_\lambda(X)$ are isomorphic to the rings $I^*(S^{[4]})$ and $I^*_\delta(S^{[4]})$ since the action of G_X is transitive on rays, but to construct an explicit isomorphism, we must find a geometric basis. To do this, we need to understand the products of Hodge classes.

3.1. **Computation of the Fujiki constants for $S^{[4]}$.** Let X be smooth variety of dimension n, and μ a partition of a nonnegative integer $|\mu|$ (we allow the empty partition of 0). To each μ we can associate a Chern monomial $\mathrm{c}_\mu(X) = \prod_{i=1}^k \mathrm{c}_k^{\mu_k}(X)$. Given a formal power series $\varphi(x) \in \mathbb{Q}[[x]]$, define the associated genus

$$\varphi(X) = \prod_i \varphi(x_i) \in H^*(X, \mathbb{Q})$$

where the x_i are the Chern roots of the tangent bundle TX. Taking the universal formal power series

$$\Phi(x) = 1 + a_1 x + a_2 x^2 + \cdots \in \mathbb{Q}[a_1, a_2, \ldots][[x]]$$

we define the universal genus $\Phi(X)$ of any smooth variety as an element of $H^*(X, \mathbb{Q})[a_1, a_2 \ldots]$. $\Phi(X)$ is a universal formal power series in the Chern classes $\mathrm{c}_1, \mathrm{c}_2, \ldots$ with coefficients polynomials in $a_1, a_2, \ldots$. In particular, taking $a_1 = 1$ and $a_i = 0$ for $i > 1$, we get the total Chern class. We will only need the universal genus for vanishing odd Chern classes; the reader may find the expansion of Φ in this case up to degree 16 in the appendix.

Let S be a smooth surface, $\varphi(x) \in \mathbb{Q}[[x]]$ a formal power series in x. Recall that $\mathcal{O}^{[n]}$ is the push-forward of the structure sheaf of the universal subscheme $Z \subset S \times S^{[n]}$ to $S^{[n]}$, and that $\det \mathcal{O}^{[n]} = -\delta$. A result of [EGL01, Theorem 4.2] implies that there are universal formal power series $A(z), B(z)$ in z such that

$$\sum_{n \geq 0} z^n \int_{S^{[n]}} \exp(\det \mathcal{O}^{[n]}) \varphi(S^{[n]}) = A(z)^{\mathrm{c}_1(S)^2} B(z)^{\mathrm{c}_2(S)}$$

for any smooth surface S. Let

$$\mathbf{F}_S(z) = \sum_{n \geq 0} z^n \int_{S^{[n]}} \exp(\det \mathcal{O}^{[n]}) \Phi(S^{[n]}) \in \mathbb{Q}[a_1, a_2, \ldots][[z]]$$

and let $\mathbf{A}(z), \mathbf{B}(z) \in \mathbb{Q}[a_1, a_2, \ldots][[z]]$ be the universal power series associated to Φ. $\mathbf{F}_{\mathbb{P}^2}(z) = \mathbf{A}(z)^9 \mathbf{B}(z)^3$ and $\mathbf{F}_{\mathbb{P}^1 \times \mathbb{P}^1}(z) = \mathbf{A}(z)^8 \mathbf{B}(z)^4$ can be easily computed by routine equivariant localization and therefore one can compute $\mathbf{A}(z), \mathbf{B}(z)$; see the appendix for a brief summary of the computation. Since $\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^2$ generate the cobordism ring, this determines $\mathbf{F}_S(z)$ for a $K3$ surface S, and in particular we can compute all products

$$\int_{S^{[n]}} \delta^{2k} \mathrm{c}_\mu(S^{[n]}) \tag{9}$$

By the following result of Fujiki, (9) determines all products of the form

$$\int_X f^{2k} \mathrm{c}_\mu(X)$$

for arbitrary $f \in H^2(X, \mathbb{Q})$:

Theorem 3.2. [Fuj87] *For X an irreducible holomorphic symplectic variety of dimension n and μ an even partition of an integer $|\mu|$, there are rational constants $\gamma_X(\mu)$ such that, for any class $f \in H^2(X, \mathbb{Z})$,*

$$\int_X f^{2k} \mathrm{c}_\mu(X) = \gamma_X(\mu) \cdot (f, f)^k, \quad \textit{for} \quad 2k = 2n - |\mu|$$

Moreover, the constant $\gamma_X(\mu)$ is a deformation invariant.

Of course, if $|\mu| > \dim X$, we have $\gamma_X(\mu) = 0$. Also, because X is holomorphic symplectic, all odd Chern classes $\mathrm{c}_i(X)$ vanish, so we require μ to be an even partition. We collect here the Fujiki constants $\gamma(\mu)$ for $n = 4$ for reference:

Corollary 3.3. *For X of $K3^{[4]}$-type, we have*

$\gamma_X(2^4) = 1992240$	$\gamma_X(2^3) = 59640$	$\gamma_X(2^2) = 4932$	$\gamma_X(2^1) = 630$	$\gamma_X(\varnothing) = 105$
$\gamma_X(2^2 4^1) = 813240$	$\gamma_X(2^1 4^1) = 24360$	$\gamma_X(4^1) = 2016$		
$\gamma_X(2^1 6^1) = 182340$	$\gamma(6^1) = 5460$			
$\gamma_X(8^1) = 25650$				
$\gamma_X(4^2) = 332730$				

Proof. This follows from the deformation invariance and the degree 4 part of

$$\mathbf{F}_S(z) = \mathbf{B}(z)^{24}$$

for S a $K3$ surface. Note that $(\delta, \delta) = -6$. $\square$

Remark 3.4. The first column of numbers are the Chern numbers of X, and were computed in [EGL01]; $\gamma_X(\varnothing)$ is the ordinary Fujiki constant. The authors are unaware of a computation of the middle three columns in the literature.

3.5. **Generalized Fujiki constants.** Let X be of $K3^{[n]}$-type. In general, for η a Hodge class, an integral of the form $\int_X f^{2k}\eta$ must be compatible with the G_X action, and therefore will be a rational multiple of $(f,f)^k$. For η a product of a power of θ and a Chern monomial, these ratios are determined by the Fujiki constants of the previous section.

Define an augmented partition (ℓ, μ) to be a partition μ of a nonnegative integer $|\mu|$ and a nonnegative integer ℓ. Set

$$|(\ell, \mu)| = 2\ell + |\mu|$$

Proposition 3.6. *For X of $K3^{[n]}$-type, $n > 1$, and (ℓ, μ) an augmented even partition, there is a rational constant $\gamma_X(\ell, \mu)$ such that for any $f \in H^2(X, \mathbb{Z})$,*

$$\int_X f^{2k}\theta^\ell \mathrm{c}_\mu(X) = \gamma_X(\ell, \mu) \cdot (f,f)^k, \quad \textit{for} \quad 2k = 2n - 2\ell - |\mu|$$

Furthermore, there are rational constants $\alpha(k, \ell)$ independent of X such that

$$\gamma_X(\ell, \mu) = \alpha(k, \ell)\gamma_X(\mu), \quad \textit{for} \quad 2k = 2n - 2\ell - |\mu|$$

Again, $\gamma_X(k, \ell, \mu) = 0$ if $|(\ell, \mu)| > \dim X$.

Proof. As mentioned above, the interesting part is the existence of the α. Let x_i be an orthonormal basis of $H^2(X, \mathbb{C})$ with respect to the Beauville–Bogomolov form. Note that $\theta = \sum_i x_i^2$. It suffices to consider the case $f = \sum_i x_i$, which has $(f,f) = 23$. Let

$$p^k(a) = \left(\sum_i a_i x_i\right)^k$$

for $a \in \mathbb{Q}^{23}$. The $p^k(a)$ span the space of degree k polynomials in x_i, so their symmetrizations

$$\overline{p}^k(a) = \frac{1}{23!}\sum_{\sigma \in S_{23}} \left(\sum_i a_i x_{\sigma(i)}\right)^k$$

span the space of degree k symmetric functions in x_i. We can therefore write

$$f^{2k}\theta^\ell = \sum_{a(k,\ell)} \lambda_{a(k,\ell)} \overline{p}^{2k+2\ell}(a(k,\ell))$$

where the sum is over finitely many $a(k,\ell)$. This expression has no dependence on the dimension of X. We have

$$\begin{aligned}\int_X f^{2k}\theta^\ell c_\mu(X) &= \frac{1}{23!}\sum_{a(k,\ell)}\lambda_{a(k,\ell)}\sum_{\sigma\in S_{23}}\int_X\left(\sum_i a(k,\ell)_i x_{\sigma(i)}\right)^{2k+2\ell} c_\mu(X)\\ &= \frac{1}{23!}\sum_{a(k,\ell)}\lambda_{a(k,\ell)}\sum_{\sigma\in S_{23}}\left(\sum_i a(k,\ell)_i x_{\sigma(i)}, \sum_i a(k,\ell)_i x_{\sigma(i)}\right)^{k+\ell}\gamma_X(\mu)\\ &= \left(\sum_{a(k,\ell)}\lambda_{a(k,\ell)}\left(\sum_i a(k,\ell)_i^2\right)^{k+\ell}\right)\gamma_X(\mu)\\ &= \alpha(k,\ell)\gamma_X(\mu)(f,f)^{k+\ell}\end{aligned}$$

where

$$\alpha(k,\ell) = \frac{1}{23^k}\sum_{a(k,\ell)}\lambda_{a(k,\ell)}\left(\sum_i a(k,\ell)_i^2\right)^{k+\ell}$$

□

Explicitly,

$$\int_X \theta c_\mu(X) = \sum_i\int_X x_i^2 c_\mu(X) = \sum_i (x_i,x_i)\gamma_X(\mu) = 23\cdot\gamma_X(\mu)$$

so $\alpha(0,1)=23$. Less trivially,

$$\begin{aligned}\int_X \theta^2 c_\mu(X) &= \int_X\left(\sum_i x_i^2\right)^2 c_\mu(X)\\ &= \int_X\left(\frac{1}{6}\sum_{i<j}(x_i+x_j)^4 + \frac{1}{6}\sum_{i<j}(x_i-x_i)^4 - \frac{19}{3}\sum_i x_i^4\right)c_\mu(X) = \frac{575}{3}\cdot\gamma_X(\mu)\end{aligned}$$

The relevant values of the α constants can be computed from Lemma 2.11 and by reducing to the $K3^{[3]}$-type case:

Lemma 3.7. *We have*

$$\begin{array}{llll}\alpha(0,1)=23 & \alpha(1,1)=\frac{25}{3} & \alpha(2,1)=\frac{27}{5} & \alpha(3,1)=\frac{29}{7}\\ \alpha(0,2)=\frac{575}{3} & \alpha(1,2)=45 & \alpha(2,2)=\frac{783}{35} & \\ \alpha(0,3)=1035 & \alpha(1,3)=\frac{1305}{7} & & \\ \alpha(0,4)=\frac{30015}{7} & & & \end{array}$$

Of course, $\alpha(k,0)=1$ for any k.

Proof. $\alpha(3,1),\alpha(2,2),\alpha(1,3),\alpha(0,4)$ are all determined by Lemma 2.11, using $\gamma_{S^{[4]}}(\varnothing) = 105$. Because $\alpha(k,\ell)$ is independent of the dimension of X, we can determine the remaining α constants from the computations of [HHT] in the $K3^{[3]}$-type cases, where

$$\begin{aligned}(\theta_{S^{[3]}})^3 &= 15525 = 1035\cdot\gamma_{S^{[3]}}(\varnothing)\\ (\delta_{S^{[3]}})^2(\theta_{S^{[3]}})^2 &= -2700 = 45\cdot(\delta_{S^{[3]}},\delta_{S^{[3]}})\cdot\gamma_{S^{[3]}}(\varnothing)\\ (\delta_{S^{[3]}})^4(\theta_{S^{[3]}}) &= 1296 = \frac{27}{5}\cdot(\delta_{S^{[3]}},\delta_{S^{[3]}})^2\cdot\gamma_{S^{[3]}}(\varnothing)\\ (\theta_{S^{[3]}})^2 c_2(S^{[3]}) &= 20700 = \frac{575}{3}\cdot\gamma_{S^{[3]}}(2^1)\\ (\delta_{S^{[3]}})^2(\theta_{S^{[3]}})c_2(S^{[3]}) &= -3600 = \frac{25}{3}\cdot(\delta_{S^{[3]}},\delta_{S^{[3]}})\cdot\gamma_{S^{[3]}}(2^1)\end{aligned}$$

since $\gamma_{S^{[3]}}(\varnothing) = 15, \gamma_{S^{[3]}}(2^1) = 108$ and $\mathrm{c}_2(S^{[3]}) = \frac{4}{3}\theta_{S^{[3]}}$.

$\square$

3.8. **A geometric basis.** $I^8(X)$ is 3-dimensional, so we expect there to be a relation among $\theta^2, \theta\mathrm{c}_2(X), \mathrm{c}_2(X)^2, \mathrm{c}_4(X)$:

Lemma 3.9. *For X of $K3^{[4]}$-type,*

$$\theta^2 = \frac{7}{5}\theta c_2 - \frac{31}{60}c_2^2 + \frac{1}{15}c_4 \tag{10}$$

Proof. Using the results of the previous section, we know the intersection form restricted to $I^8(X)$ in terms of the basis $\theta^2, \theta\mathrm{c}_2(X), \mathrm{c}_2(X)^2, \mathrm{c}_4(X)$:

$$\begin{pmatrix} 450225 & 1035\cdot 630 & \frac{575}{3}\cdot 4932 & \frac{575}{3}\cdot 2016 \\ 1035\cdot 630 & \frac{575}{3}\cdot 4932 & 23\cdot 59640 & 23\cdot 24360 \\ \frac{575}{3}\cdot 4932 & 23\cdot 59640 & 1992240 & 813240 \\ \frac{575}{3}\cdot 2016 & 23\cdot 24360 & 813240 & 332730 \end{pmatrix} \tag{11}$$

As expected, the matrix is rank 3. By Poincaré duality, a generator of the kernel gives the relation. $\square$

Corollary 3.10. $\mathrm{c}_2(S^{[4]}) = 3Z + 33Y - W$

Proof. Suppose $\mathrm{c}_2(S^{[4]}) = wW + xX + yY + zZ$ for $w, x, y, z \in \mathbb{Q}$. Taking the product with $\theta^3, \delta^2\theta^2, \delta^4\theta, \delta^4$ yields the equation

$$\begin{pmatrix} -6075 & -2700 & \frac{30375}{2} & \frac{96525}{2} \\ 15066 & 6696 & -3213 & -16335 \\ -19116 & -8496 & 1854 & 14058 \\ 29160 & 12960 & -1620 & -17820 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 652050 \\ -170100 \\ 122472 \\ -136080 \end{pmatrix}$$

The matrix has rank 2. Computing generators of the kernel, we can write

$$\mathrm{c}_2(S^{[4]}) = \left(-\frac{4}{9}u - \frac{4}{27}v\right)W + \left(u - \frac{21}{4}\right)X + (v+42)Y - \frac{v}{3}Z$$

Similarly, computing $\mathrm{c}_2(S^{[4]})^2$ and intersecting with $\theta^2, \delta^2\theta, \delta^4$ yields 3 equations: $945300 = \theta^2\mathrm{c}_2(S^{[4]})^2$, $-246600 = \theta\delta^2\mathrm{c}_2(S^{[4]})$ and $177552 = \delta^4\mathrm{c}_2(S^{[4]})$ which have exactly two common solutions: $(u, v) = (\frac{21}{4}, -9), (\frac{497}{116}, -\frac{285}{29})$. Finally, only one of these solutions, $(u, v) = (\frac{21}{4}, -9)$, satisfies the additional equation $1992240 = \mathrm{c}_2(S^{[4]})^4$, and this gives the desired equation. $\square$

Recall that $I^4(X)$ is 2-dimensional, whereas $I^4_\lambda(X)$ is 4-dimensional. We already have $\lambda^2 \in I^4_\lambda(X)$. We need one more geometrically defined class in $I^4_\lambda(X)$ independent from λ^2 and $I^4(X)$ to get a basis for $I^4_\lambda(X)$:

Definition 3.11. Given a class $\lambda \in H^2(X, \mathbb{Q})$ (with $(\lambda, \lambda) \neq 0$ so no power of λ is zero), define $\alpha \in I^4_\lambda(X)$ by Poincaré duality to be the unique class (up to a multiple) that intersects trivially with λ^6 and $I^{12}(X)$.

Lemma 3.12. *For $X = S^{[4]}$ and $\lambda = \delta$, we may take $\alpha = X - 3Y + Z$ which intersects trivially with*

$$\delta^4\theta, \delta^4\mathrm{c}_2(S^{[4]}), \delta^2\theta^2, \delta^2\theta\mathrm{c}_2(S^{[4]}), \delta^2\mathrm{c}_2(S^{[4]})^2, \theta^3, \theta^2\mathrm{c}_2(S^{[4]}), \theta\mathrm{c}_2(S^{[4]})^2, \mathrm{c}_2(S^{[4]})^3$$

Further, $\alpha^2\theta^2 = 9450$, $\alpha^2\theta\mathrm{c}_2(S^{[4]}) = 14148$ and $\alpha^2\mathrm{c}_2(S^{[4]})^2 = 21168$.

Proof. By intersecting with θ and $\mathrm{c}_2(S^{[4]})$ using Corollary 3.3 and Lemma 3.7, we see that θ^3 and $\theta^2\mathrm{c}_2(S^{[4]})$ are independent in $I^{12}(S^{[4]})$, so it is enough to show that α intersects these two classes to conclude it intersects trivially with each of the four degree 12 Hodge classes at the end of the list. This, along with all the other claimed products, follow from Corollary 3.10, equation (8), and our knowledge of the product structure. Indeed,

$$\begin{aligned}
\alpha^2 &= -3G + 30D + 42F + 3H + 6E \\
\alpha\delta^2 &= -18B + 162C \\
\alpha\theta &= 88D + 8E - 27C + 3B - 88F + 20G + 4H \\
\alpha \mathrm{c}_2(S^{[4]}) &= -54C + 132D + 6B + 12E + 30G + 6H - 132F \\
\theta^2 &= -8E + \frac{19}{2}H + \frac{215}{2}G - 64D - \frac{33}{4}A - \frac{97}{4}B + 1117F - \frac{873}{4}C \\
\delta^4 &= -81A - 81B - 729C - 192D - 96E + 84F + 30G + 6H \\
\theta \mathrm{c}_2(S^{[4]}) &= -\frac{27}{2}A - \frac{747}{2}C - \frac{83}{2}B - 8E + 1630F + 153G + 13H - 48D \\
\mathrm{c}_2(S^{[4]})^2 &= 18H - 8E - 69B - 8D + 218G - 21A - 621C + 2380F
\end{aligned}$$

and the pairwise products are easily computed.

□

Because the cup-product structure on $H^*(S^{[4]}, \mathbb{Z})$ is preserved under deformation, and the monodromy group acts transitively on rays in $H^2(S^{[4]}, \mathbb{Q})$, we immediately conclude the same for arbitrary λ :

Corollary 3.13. *For α chosen as in Definition 3.11 with respect to $\lambda \in H^2(X, \mathbb{Z})$, α intersects trivially with*

$$\lambda^4\theta, \lambda^4\mathrm{c}_2(X), \lambda^2\theta^2, \lambda^2\theta\mathrm{c}_2(X), \lambda^2\mathrm{c}_2(X)^2, \theta^3, \theta^2\mathrm{c}_2(X), \theta\mathrm{c}_2(X)^2, \mathrm{c}_2(X)^3$$

Further, up to a rational square, $\alpha^2\theta^2 = 9450$, $\alpha^2\theta\mathrm{c}_2(X) = 14148$ and $\alpha^2\mathrm{c}_2(X)^2 = 21168$.

3.14. **Middle cohomology.** Putting Lemma 3.7 and Corollaries 3.3, and 3.13 together, we now know the complete intersection form on middle cohomology $I^8_\lambda(X)$ with respect to the basis:

$$\lambda^4, \lambda^2\theta, \lambda^2\mathrm{c}_2(X), \theta^2, \theta\mathrm{c}_2(X), \mathrm{c}_2(X)^2, \alpha\theta, \alpha\mathrm{c}_2(X) \tag{12}$$

Denoting it by $M(\lambda)$, it is:

$$\begin{pmatrix}
105(\lambda,\lambda)^4 & 435(\lambda,\lambda)^3 & 630(\lambda,\lambda)^3 & 2349(\lambda,\lambda)^2 & 3402(\lambda,\lambda)^2 & 4932(\lambda,\lambda)^2 & & \\
435(\lambda,\lambda)^3 & 2349(\lambda,\lambda)^2 & 3402(\lambda,\lambda)^2 & 19575(\lambda,\lambda) & 28350(\lambda,\lambda) & 44110(\lambda,\lambda) & & \\
630(\lambda,\lambda)^3 & 3402(\lambda,\lambda)^2 & 4932(\lambda,\lambda)^2 & 28350(\lambda,\lambda) & 44110(\lambda,\lambda) & 59640(\lambda,\lambda) & & \\
2349(\lambda,\lambda)^2 & 19575(\lambda,\lambda) & 28350(\lambda,\lambda) & 450225 & 652050 & 945300 & & \\
3402(\lambda,\lambda)^2 & 28350(\lambda,\lambda) & 41100(\lambda,\lambda) & 652050 & 945300 & 1371720 & & \\
4932(\lambda,\lambda)^2 & 41100(\lambda,\lambda) & 59640(\lambda,\lambda) & 945300 & 1371720 & 1992240 & & \\
 & & & & & & 9450 & 14148 \\
 & & & & & & 14148 & 21168
\end{pmatrix}$$

Note that this matrix is nonsingular if $(\lambda, \lambda) \neq 0$, and therefore (12) is in fact a basis.

4. Lagrangian n-planes in X

Let X be a $2n$ dimensional holomorphic symplectic variety, and suppose that $\mathbb{P}^n \subset X$ is a smoothly embedded Lagrangian n-plane. By a simple calculation,

Lemma 4.1. [HHT]*Denote by h the hyperplane class on $\mathbb{P}^n$. Then in the above setup,*

$$\mathrm{c}_{2j}(T_X|_{\mathbb{P}^n}) = (-1)^j h^{2j} \binom{n+1}{j}$$

Proof. We have

$$0 \to T_{\mathbb{P}^n} \to T_X|_{\mathbb{P}^n} \to N_{\mathbb{P}^n/X} \to 0$$

and since $\mathbb{P}^n$ is Lagrangian, $N_{\mathbb{P}^n/X} \cong T^*_{\mathbb{P}^n}$, so

$$\mathrm{c}(T_X|_{\mathbb{P}^n}) = (1+h)^{n+1}(1-h)^{n+1} = (1-h^2)^{n+1}$$

□

Let θ be the Beauville–Bogomolov class. Then for $n=4$,

Lemma 4.2. $\theta|_{\mathbb{P}^4} = -\frac{7}{2}h^2$.

Proof. Let $\theta|_{\mathbb{P}^4} = nh^2$. Equation (10) implies that $60n^2 = 7\cdot 12n(-5) - 31(-5)^2 + 4(10)$ which implies the lemma. □

Finally, the last intersection theoretic piece of data we need is

$$[\mathbb{P}^4]^2 = \mathrm{c}_4(N_{\mathbb{P}^4/X}) = \mathrm{c}_4(T^*_{\mathbb{P}^4}) = 5 \tag{13}$$

since $\mathbb{P}^4$ is Lagrangian.

Assume now that X is deformation equivalent to a Hilbert scheme of 4 points on a $K3$ surface. Let $\ell \in H_2(X,\mathbb{Z})$ be the class of the line, and $\lambda = 6\ell \in H^2(X,\mathbb{Z})$, via the embedding $H^2(X,\mathbb{Z}) \subset H_2(X,\mathbb{Z})$ induced by the Beauville–Bogomolov form. Note that $\lambda|_{\mathbb{P}^4} = \frac{(\lambda,\lambda)}{6}h$ since $\langle \lambda|_{\mathbb{P}^4}, \ell\rangle = \langle \lambda, \ell\rangle = \frac{1}{6}(\lambda,\lambda)$ by the definition of λ. Then

$$[\mathbb{P}^4] = a\lambda^4 + b\lambda^2\theta + c\lambda^2\mathrm{c}_2(X) + d\theta^2 + e\theta\mathrm{c}_2(X) + f\mathrm{c}_2(X)^2 + g\theta\alpha + h\mathrm{c}_2(X)\alpha$$

Assume that $\alpha|_{\mathbb{P}^4} = yh^2$, for $y \in \mathbb{Q}$. Intersecting this class with each of (12),

$$\lambda^4, \lambda^2\theta, \lambda^2\mathrm{c}_2(X), \theta^2, \theta\mathrm{c}_2(X), \mathrm{c}_2(X)^2, \alpha\theta, \alpha\mathrm{c}_2(X)$$

yields by Lemmas 4.1 and 4.2 the equation

$$M(\lambda)[\mathbb{P}^4] = \begin{pmatrix} \left(\frac{(\lambda,\lambda)}{6}\right)^4 \\ -\frac{7}{2}\left(\frac{(\lambda,\lambda)}{6}\right)^2 \\ -5\left(\frac{(\lambda,\lambda)}{6}\right)^2 \\ \frac{49}{4} \\ \frac{35}{2} \\ 25 \\ -\frac{7}{2}y \\ -5y \end{pmatrix} \tag{14}$$

from which it follows that

$$[\mathbb{P}^4] = \begin{pmatrix} \frac{1}{608256}\left(25 + \frac{700}{(\lambda,\lambda)} + \frac{1764}{(\lambda,\lambda)^2}\right) \\ -\frac{1}{2737152}\left(25(\lambda,\lambda) + 3276 + \frac{15876}{(\lambda,\lambda)}\right) \\ \frac{1}{38016}\left(23 + \frac{126}{(\lambda,\lambda)}\right) \\ \frac{1}{5474304}\left((\lambda,\lambda)^2 + 252(\lambda,\lambda) - 41148\right) \\ -\frac{1}{190080}\left(5(\lambda,\lambda) - 2142\right) \\ -\frac{1}{240} \\ \frac{31y}{1188} \\ -\frac{7y}{396} \end{pmatrix} \tag{15}$$

Finally, (13) yields:

$$5 = \frac{25}{788299776}x^4 + \frac{175}{98537472}x^3 + \frac{403}{10948608}x^2 - \frac{7}{2376}y^2 + \frac{7}{33792}x + \frac{65}{67584}$$

where $x = (\lambda, \lambda)$. This may be rewritten as

$$y^2 = \frac{5^2}{2^{12} \cdot 3^4 \cdot 7}x^4 + \frac{5^2}{2^9 \cdot 3^4}x^3 + \frac{13 \cdot 31}{2^9 \cdot 3^2 \cdot 7}x^2 + \frac{3^2}{2^7}x - \frac{3^2 \cdot 5 \cdot 7^2 \cdot 197}{2^8} \tag{16}$$

Note that while we may have $y \in \mathbb{Q}$, x must be integral. Also note that there is a solution compatible with Conjecture 1, namely $(x, y) = (-126, 0)$. By the analysis of the next section,

Proposition 4.3. *The only solution of (16) with $x \in \mathbb{Z}$ and $y \in \mathbb{Q}$ is $(x, y) = (-126, 0)$.*

It then follows that

Theorem 4.4. *Let X be of $K3^{[4]}$-type, $\mathbb{P}^4 \subset X$ be a smoothly embedded Lagrangian 4-plane, $\ell \in H_2(X, \mathbb{Z})$ the class of a line in $\mathbb{P}^4$, and $\rho = 2\ell \in H^2(X, \mathbb{Q})$. Then ρ is integral, and*

$$[\mathbb{P}^4] = \frac{1}{337920}\left(880\rho^4 + 1760\rho^2 \mathrm{c}_2(X) - 3520\theta^2 + 4928\theta \mathrm{c}_2(X) - 1408\mathrm{c}_2(X)^2\right) \tag{17}$$

Further, we must have $(\ell, \ell) = -\frac{7}{2}$.

Proof. (17) is obtained from (15) by substituting $(\lambda, \lambda) = -126$ and $y = 0$, after setting $\rho = \frac{1}{3}\lambda$. It remains to show that ρ is integral. Following [HHT], after deforming to a Hilbert scheme of points on a $K3$ surface S, we can write

$$\ell = D + m\delta^\vee$$

using the decomposition dual to (1), for $D \in H_2(S, \mathbb{Z})$. Since

$$(\ell, \ell) = D^2 - \frac{m^2}{6} = -\frac{7}{2}$$

and $D^2 \in 2\mathbb{Z}$, $3|m$. For 2ℓ to be an integral class in $H^2(X, \mathbb{Z})$, by Poincaré duality it is sufficient for the form $(2\ell, \cdot)$ on $H_2(X, \mathbb{Z})$ to be integral, which it obviously is, since $(\delta^\vee, \delta^\vee) = -\frac{1}{6}$. $\square$

5. Solving the Diophantine equation

The Diophantine equation (16) to solve is

$$y^2 = \frac{5^2}{2^{12} \cdot 3^4 \cdot 7}x^4 + \frac{5^2}{2^9 \cdot 3^4}x^3 + \frac{13 \cdot 31}{2^9 \cdot 3^2 \cdot 7}x^2 + \frac{3^2}{2^7}x - \frac{3^2 \cdot 5 \cdot 7^2 \cdot 197}{2^8}$$

with $x \in \mathbb{Z}$ and $y \in \mathbb{Q}$. Let $\mathcal{C}$ be the affine curve described by the equation. After the change of variables $(x_1, y_1) = (x + 126, 2^6 \cdot 3^2 \cdot 7y)$, every point $(x, y) \in \mathcal{C}$ with $x \in \mathbb{Z}$ gives an integral point (x_1, y_1) on the curve $\mathcal{C}_1$:

$$y_1^2 = (5^2 \cdot 7)x_1^4 - (2^6 \cdot 5^2 \cdot 7^2)x_1^3 + (2^7 \cdot 3^2 \cdot 7 \cdot 23 \cdot 71)x_1^2 - (2^{11} \cdot 3^4 \cdot 7^2 \cdot 11^2)x_1$$

Lemma 5.1. *For an integer v consider the elliptic curve $\mathcal{E}_v$ given by the Weierstrass equation*

$$y_2^2 = x_2^3 - (2^6 \cdot 5^2 \cdot 7^2 \cdot v)x_2^2 + (2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 23 \cdot 71 \cdot v^2)x_2 - (2^{11} \cdot 3^4 \cdot 5^4 \cdot 7^4 \cdot 11^2 \cdot v^3)$$

Then every integral point $(x_1, y_1) \neq (0, 0)$ on the curve $\mathcal{C}_1$ corresponds to an integral point (x_2, y_2) on one of the curves $\mathcal{E}_v$ where

$$x_1 = u^2 v \qquad x_2 = 5^2 \cdot 7 \cdot v^2 u^2$$
$$y_1 = uvw \qquad y_2 = 5^2 \cdot 7 \cdot v^2 w$$

for some integers u, v, w where v is a divisor of $2 \cdot 3 \cdot 7 \cdot 11$.

Proof. Certainly if $x_1 = 0$ then $y_1 = 0$ and it can be checked that if $y_1 = 0$ then $x_1 = 0$ is the only rational solution. So let us assume for the remaining that $x_1, y_1 \neq 0$. Note that since $x_1 \in \mathbb{Z}$ it follows that $y_1 \in \mathbb{Z}$ and $x_1 \mid y_1^2$. Since $x_1, y_1 \neq 0$ we may write $x_1 = u^2 v$ and $y_1 = uvw$ for $u, v, w \in \mathbb{Z}$ with v square-free. Rewriting the equation we get

$$vw^2 = 5^2 \cdot 7 \cdot u^6 v^3 - 2^6 \cdot 5^2 \cdot 7^2 \cdot u^4 v^2 + 2^7 \cdot 3^2 \cdot 7 \cdot 23 \cdot 71 \cdot u^2 v - 2^{11} \cdot 3^4 \cdot 7^2 \cdot 11^2$$

and we conclude that v is a divisor of $2 \cdot 3 \cdot 7 \cdot 11$.

Multiplying by $5^4 \cdot 7^2 \cdot v^3$ and making the change of variables $y_2 = 5^2 \cdot 7 \cdot v^2 \cdot w$ and $x_2 = 5^2 \cdot 7 \cdot v^2 \cdot u^2$ we get the equation

$$\begin{aligned}(5^2 \cdot 7 \cdot v^2 \cdot w)^2 = {} & (5^2 \cdot 7 \cdot v^2 \cdot u^2)^3 - 2^6 \cdot 5^2 \cdot 7^2 \cdot v \cdot (5^2 \cdot 7 \cdot v^2 \cdot u^2)^2 \\ & + 2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 23 \cdot 71 \cdot v^2 (5^2 \cdot 7 \cdot v^2 \cdot u^2) - 2^{11} \cdot 3^4 \cdot 5^4 \cdot 7^4 \cdot 11^2 \cdot v^3\end{aligned}$$

which yields

$$y_2^2 = x_2^3 - 2^6 \cdot 5^2 \cdot 7^2 \cdot v \cdot x_2^2 + 2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 23 \cdot 71 \cdot v^2 x_2 - 2^{11} \cdot 3^4 \cdot 5^4 \cdot 7^4 \cdot 11^2 \cdot v^3$$

and therefore a point $(x_2, y_2) \in \mathcal{E}_v(\mathbb{Z})$. □

Thus to find the required points on $\mathcal{C}$ we need to find the integral solutions of the elliptic curve $\mathcal{E}_v$ above whenever v is a divisor of $2 \cdot 3 \cdot 7 \cdot 11$, of which there are 32 (positive and negative).

Lemma 5.2. *Suppose v is a divisor of $2 \cdot 3 \cdot 7 \cdot 11$ such that $7 \nmid v$. If the curve $\mathcal{E}_v$ has an integral solution $(5^2 \cdot 7 \cdot u^2 v^2, 5^2 \cdot 7 \cdot v^2 w)$ then $v \in \{-1, -2, -11, -22\}$.*

Proof. Note from the equation

$$vw^2 = 5^2 \cdot 7 \cdot u^6 v^3 - 2^6 \cdot 5^2 \cdot 7^2 \cdot u^4 v^2 + 2^7 \cdot 3^2 \cdot 7 \cdot 23 \cdot 71 \cdot u^2 v - 2^{11} \cdot 3^4 \cdot 7^2 \cdot 11^2$$

we deduce that $7 \mid vw^2$. Since $7 \nmid v$ it follows that $7 \mid w$ so it must be that $5^2 u^6 v^3 + 2^7 \cdot 3^2 \cdot 23 \cdot 71 u^2 v \equiv 0 \pmod 7$ in other words $u^2 v \equiv 3u^6 v^3 \pmod 7$. Since v is invertible we get $5u^2 \equiv u^6 v^2$. If $7 \nmid u$ then we would have that 5 is a quadratic residue mod 7, which is not true. So $7 \mid u$. Rewriting the equation for $w = 7w_1$ and $u = 7u_1$ we get

$$vw_1^2 = 5^2 \cdot 7^5 \cdot u_1^6 v^3 - 2^6 \cdot 5^2 \cdot 7^4 \cdot u_1^4 v^2 + 2^7 \cdot 3^2 \cdot 7 \cdot 23 \cdot 71 \cdot u_1^2 v - 2^{11} \cdot 3^4 \cdot 11^2$$

so necessarily $vw_1^2 \equiv 3 \pmod 7$. But the only square-free divisors v of $2 \cdot 3 \cdot 11$ for which such w_1 exist are $3, 6, 33, 66, -1, -2, -11, -22$.

If $3 \mid v$ then we could write $v = 3v_1$ so we would get

$$v_1 w_1^2 = 5^2 \cdot 3^2 \cdot 7^5 \cdot u_1^6 v_1^3 - 2^6 \cdot 3 \cdot 5^2 \cdot 7^4 \cdot u_1^4 v_1^2 + 2^7 \cdot 3^2 \cdot 7 \cdot 23 \cdot 71 \cdot u_1^2 v_1 - 2^{11} \cdot 3^3 \cdot 11^2$$

which would imply that $3 \mid v_1 w_1^2$. Since $3 \nmid v_1$ (as v is square-free) it follows that $3^2 \mid v_1 w_1^2$ but then 3^2 divides the right hand side so we deduce that $3 \mid u_1$. Writing $w_1 = 3w_2$ and $u_1 = 3u_2$ we get

$$v_1 w_2^2 = 5^2 \cdot 3^6 \cdot 7^5 \cdot u_2^6 v_1^3 - 2^6 \cdot 3^3 \cdot 5^2 \cdot 7^4 \cdot u_2^4 v_1^2 + 2^7 \cdot 3^2 \cdot 7 \cdot 23 \cdot 71 \cdot u_2^2 v_1 - 2^{11} \cdot 3 \cdot 11^2$$

As before, we get that $3^2 \mid v_1 w_2^2$ but now 3^2 cannot divide the right hand side.

The remaining possibilities for v are $-1, -2, -11, -22$. □

Lemma 5.3. *If the curve $\mathcal{E}_v$ where v is a divisor of $2 \cdot 3 \cdot 7 \cdot 11$ such that $7 \mid v$ has an integral solution $(5^2 \cdot 7 \cdot u^2 v^2, 5^2 \cdot 7 \cdot v^2 w)$ then $v \in \{7, 14, 77, 154\}$.*

Proof. Writing $v = 7v_1$ we get

$$v_1 w^2 = 5^2 \cdot 7^3 \cdot u^6 v_1^3 - 2^6 \cdot 5^2 \cdot 7^3 \cdot u^4 v_1^2 + 2^7 \cdot 3^2 \cdot 7 \cdot 23 \cdot 71 \cdot u^2 v_1 - 2^{11} \cdot 3^4 \cdot 7 \cdot 11^2$$

Since v is square-free $7 \nmid v_1$ so we deduce that $7 \mid w$. Writing $w = 7w_1$ we get

$$7v_1 w_1^2 = 5^2 \cdot 7^2 \cdot u^6 v_1^3 - 2^6 \cdot 5^2 \cdot 7^2 \cdot u^4 v_1^2 + 2^7 \cdot 3^2 \cdot 23 \cdot 71 \cdot u^2 v_1 - 2^{11} \cdot 3^4 \cdot 11^2$$

which implies that $u^2 v_1 \equiv 4 \pmod 7$. The only v_1 among the square-free divisors of $2 \cdot 3 \cdot 11$ for which such u exist are $1, 2, 11, 22, -3, -6, -33, -66$ giving $v \in \{7, 14, 77, 154, -21, -42, -231, -462\}$.

As in the previous lemma, under the assumption that $3 \mid v$ we get a contradiction. The remaining possibilities are $v \in \{7, 14, 77, 154\}$. $\square$

Six of the eight cases to which we've reduced in Lemmas 5.2 and 5.3 are then treated directly by:

Lemma 5.4. *If $v \in \{-1, -2, 7, 14, 77, 154\}$ the curve $\mathcal{E}_v$ has no integral points of the form $(5^2 \cdot 7 \cdot u^2 v^2, 5^2 \cdot 7 \cdot v^2 w)$.*

Proof. We compute the integral points of these elliptic curves using Sage ([S+13]) version 5.2 run on William Stein's cluster `geom.math.washington.edu` and collect the results below. The general method is by finding a basis for the Mordell–Weil group of a rational elliptic curve (using the command `gens` in Sage) and then finding a list of all the integral points using this basis (using the command `integral_points(mw_basis=...)` in Sage). Typically the computation of a basis is very difficult computationally (on the order of hours for the curves under consideration), whereas the computation of integral points is quite fast (on the order of seconds). As such we include bases for the Mordell–Weil groups of these elliptic curves in which case the computation of integral points can be reproduced quickly.

	Mordell–Weil basis, runtime	Integral points
$\mathcal{E}_7$	$(\frac{23929444}{81}, \frac{22042862072}{729})$, 22 seconds	$\varnothing$
$\mathcal{E}_{77}$	$(\frac{1427771448857345912 04}{47183614355089}, \frac{511502202996707134646435200 08}{324105804064380058937})$, 405 minutes	$\varnothing$
$\mathcal{E}_{154}$	$(\frac{267909856900}{23409}, -\frac{745374319856306 00}{3581577})$, 400 minutes	$\varnothing$
$\mathcal{E}_{-2}$	$(-\frac{40566784}{529}, \frac{27776430464}{12167})$ $(-\frac{3296728575}{65536}, -\frac{114720819732225}{16777216})$, 58 minutes	$\varnothing$
$\mathcal{E}_{-1}$	$(-27900, 2266200)$ $(\frac{138825}{4}, \frac{125561925}{8})$, 80 minutes $(166980, 85186200)$	$(-39196, \pm 156792)$ $(-27900, \pm 2266200)$ $(166980, \pm 85186200)$
$\mathcal{E}_{14}$	$(564480, 49392000)$ $(940800, 451113600)$ $(1317120, 945033600)$, 15 seconds $(2257920, 2617776000)$	$(564480, \pm 49392000)$ $(604905, \pm 101433675)$ $(632100, \pm 129859800)$ $(683844, \pm 180931128)$ $(755825, \pm 251976375)$ $(940800, \pm 451113600)$ $(1063680, \pm 599510400)$ $(1317120, \pm 945033600)$ $(1361220, \pm 1010272200)$ $(2257920, \pm 2617776000)$ $(3066624, \pm 4451914368)$ $(3327780, \pm 5110549800)$ $(11863929, \pm 38995732083)$ $(12603780, \pm 42818542200)$ $(13848576, \pm 49513570176)$ $(72195620, \pm 608777597400)$ $(1964277504, \pm 87032792472192)$

None of the integral points (x, y) on $\mathcal{E}_v$ have the required form $x = 5^2 \cdot 7 \cdot u^2 v^2$, and the proof is concluded. $\square$

The remaining two curves $\mathcal{E}_{-11}, \mathcal{E}_{-22}$ are computationally less tractable. The standard computation of generators for the Mordell–Weil group in Sage for these two elliptic curves does not terminate in any reasonable time, though the closed-source algebra system Magma ([BCP97]) allows one to perform a reasonably fast analysis of these two elliptic curves. We will give two computational proofs that these curves do not have integral points of the required type: the first, in the open source Sage, relies on Kolyvagin's proof of the Birch and Swinnerton-Dyer conjecture of elliptic curves over $\mathbb{Q}$ of analytic rank 1 while the second, in the proprietary Magma, uses a two descent procedure, and is given mainly as a corroboration of the results from Sage. We are greatful to Michael Stoll for explaining how to do the computations in Magma. We remark that the same methods will in principle work for the other curves in Lemma 5.4 of rank 1, namely $\mathcal{E}_{77}$ and $\mathcal{E}_{154}$.

We first need the following lemma.

Lemma 5.5. *If E is one of the curves $\mathcal{E}_{-11}$ and $\mathcal{E}_{-22}$ then $L'(E,1) \neq 0$.*

Proof. We recall a result of Cohen ([Coh93, 5.6.12]) that

$$L'(E,1) = 2\sum_{n\geq 1} \frac{a_n}{n} E_1\left(\frac{2\pi n}{\sqrt{N}}\right)$$

where N is the conductor of E and $E_1(x) = \int_1^\infty e^{-xy}y^{-1}dy$ is the exponential integral. Truncating this series at k, one gets $L'(E,1) = L_k + \varepsilon_k$ where $L_k = 2\sum_{n=1}^{k} \frac{a_n}{n} E_1\left(\frac{2\pi n}{\sqrt{N}}\right)$ and the error is explicitly bounded $|\varepsilon_k| \leq 2e^{-2\pi(k+1)/\sqrt{N}}/(1-e^{-2\pi/\sqrt{N}})$ (for a proof see [GJP+09, §2.2]). This estimate is at the basis of the Sage command `E.lseries().deriv_at1(k)` (here k is the cutoff). In principle, if one expects that $L'(E,1) \neq 0$ then it suffices to choose the cutoff index k large enough that $|\varepsilon_k| < |L_k|$ in which case $L'(E,1)$ will be forced to be nonzero.

However, the curves under consideration have such a large conductor (in both cases $N = 83060209520534400$) that k has to be choosen on the order of $8 \cdot 10^8$, which is too large for practical purposes in Sage: in effect one runs out of memory in the computation of the coefficients a_n and $E_1(2\pi n/\sqrt{N})$. We compute the coefficients a_n for the two curves up to $k = 8 \cdot 10^8$ by first computing a_p for p prime (this operation takes about 2 hours for each curve) and then reconstructing a_n using the following: if $(m,n) = 1$ then $a_{mn} = a_m a_n$, if $p \nmid N$ then $a_{p^k} = a_p a_{p^{k-1}} - p a_{p^{k-2}}$ and if $p \mid N$ then $a_{p^k} = a_p^k$. For each curve the resulting file is on the order of $2.5GB$ and the computation takes about 3 hours for each curve. Next, we compute $E_1(2\pi n/\sqrt{N})$ for $1 \leq n \leq 8 \cdot 10^8$ (once, as the two curves have the same conductor). The command `exponential_integral_1(`$2\pi/\sqrt{N}$`, k)` in Sage should return the desired list but k is too large for this operation to be feasible. Instead, noting that Sage's `exponential_integral_1` is a wrapper for the PARI ([The12], version 2.5.4) function `veceint1`, we rewrote this PARI function to write the coefficients $E_1(2\pi n/\sqrt{N})$ to a file, instead of collecting them in a prohibitively long vector. The subsequent computation was run for about 10 hours resulting in 35GB of data.

Each coefficient $E_1(2\pi n/\sqrt{N}) = E_{1,n} + \varepsilon_{1,n}$ where $E_{1,n}$ is the number computed in PARI and $|\varepsilon_{1,n}| < 10^{-20}$ is the chosen precision. We denote by ℓ_E the value $2\sum_{n=1}^{k} \frac{a_n}{n} E_{1,n}$ computed in Sage and PARI using the cutoff $k = 8 \cdot 10^8$. Therefore we compute the value of $L'(E,1) = \ell_E + \varepsilon$ where the error is then at most (using the inquality $|a_n| \leq n$ from [GJP+09, Lemma 2.9])

$$\begin{aligned}
\varepsilon &< 2\sum_{n=1}^{k} \frac{|a_n|}{n} \cdot 10^{-20} + \varepsilon_k \\
&< 2 \cdot 10^{-20} \cdot k + \varepsilon_k \\
&< 16 \cdot 10^{-12} + \varepsilon_k \\
&< 3
\end{aligned}$$

Finally, in Sage we find $\ell_{\mathcal{E}_{-11}} = 12.561\ldots$ and $\ell_{\mathcal{E}_{-22}} = 16.069\ldots$ and the conclusion follows.

$\square$

Lemma 5.6. *If $v \in \{-11, -22\}$ the curve $\mathcal{E}_v$ has no integral points of the form $(5^2 \cdot 7 \cdot u^2v^2, 5^2 \cdot 7 \cdot v^2 w)$.*

Proof. First, suppose $E/\mathbb{Q}$ is an elliptic curve of rank 1 and $P \in E(\mathbb{Q})$ is a point of infinite order (a fact which can be checked computationally by requiring that the canonical height of the point is nonzero). We would like a fast algorithm for finding a generator P_0 of the Mordell–Weil group $E(\mathbb{Q})$. Suppose P_0 is a generator of $E(\mathbb{Q})$ in which case $P = nP_0$ for some integer n as E has rank 1. If P is not a generator then $|n| \geq 2$.

Write h for the logarithmic height and $\widehat{h}$ for the canonical logarithmic height on $E(\mathbb{Q})$. There exists a constant B, depending only on E, called the Cremona–Pricket–Siksek bound, such that for all $Q \in E(\mathbb{Q})$, $h(Q) \leq \widehat{h}(Q) + B$. Given a Weierstrass equation for E, the constant B can be computed in Sage using the command `CPS_height_bound` and in Magma using the command `SiksekBound`. If $|n| \geq 2$ then $\widehat{h}(P_0) = \dfrac{\widehat{h}(P)}{n^2} \leq \dfrac{\widehat{h}(P)}{4}$ so $h(P_0) = \widehat{h}(P_0) + h(P_0) - \widehat{h}(P_0) \leq \dfrac{\widehat{h}(P)}{4} + B$. Thus, to find P_0 one only needs to search for rational points of height at most $\dfrac{1}{4}\widehat{h}(P) + B$. One can find rational points of height $\leq h_0$ in Sage using the command `rational_points(bound=`h_0`)` and a generator P_0 can be found in the resulting finite list.

We will first check that the elliptic curves E_{-11} and E_{-22} have rank 1 and then we will apply the above described procedure to find a basis for the Mordell–Weil group. The command `DescentInformation` in Magma rapidly returns rank 1 for our curves. As mentioned above, in Sage one needs a different approach (note that the Sage command `analytic_rank` yields only the *probable* analytic rank, equal to 1, in about 17 hours for each of the two curves).

Recall Kolyvagin's result that if E is a (necessarily modular) rational elliptic curve of analytic rank 0 or 1 then the Birch and Swinnerton-Dyer conjecture is true, i.e., the rank of the elliptic curve equals its analytic rank. We will exhibit below points of infinite order on each of the two elliptic curves and so their rank (and so also their analytic rank) is at least 1. Lemma 5.5 implies that $L'(E, 1) \neq 0$ and so their analytic rank, and therefore also their rank, must be 1, as desired.

We proceed with finding bases for the Mordell-Weil groups. We start with the curve $E = \mathcal{E}_{-11}$. The elliptic curve E is

$$y^2 = x^3 + 2^6 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot x^2 + 2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 23 \cdot 71 \cdot x + 2^{11} \cdot 3^4 \cdot 5^4 \cdot 7^4 \cdot 11^5$$

Via the change of variables $x = 4x_1 - 287468, y = 8y_1$ we get the minimal Weierstrass equation E'

$$y_1^2 = x_1^3 - x_1^2 + 1933249267x_1 + 116312127942837$$

One may easily check that the point

$$P = \left(\frac{195693}{4}, \frac{144883425}{8}\right)$$

is in $E'(\mathbb{Q})$ (this point was found using Magma, but checking that it is a point on the curve is immediate without necessarily using a computer). The command `height` in Sage computes the canonical height to be $\widehat{h}(P) = 11.289\ldots$ (and so P has infinite order) while the CPS bound is $B = 11.424\ldots$.

As explained before, we seek a generator of $E'(\mathbb{Q})$. If P is not a generator then a generator will have height at most $\widehat{h}(P)/4 + B$. However, a computation in Sage shows that the only rational points with this height bound are $0, \pm P$ and so P must be a generator of $E'(\mathbb{Q})$.

Transfering back to $E(\mathbb{Q})$ one obtains the generator $(x, y) = (-91775, 144883425)$ of $E(\mathbb{Q})$. Using the command `integral_points` in Sage to compute the integral points, inputting manually the basis for $E(\mathbb{Q})$, one obtains that the only integral points of $E(\mathbb{Q})$ are $(-91775, \pm 144883425)$ but $x = -91775$ is not of the required form.

The elliptic curve $E = \mathcal{E}_{-22}$ is

$$y^2 = x^3 + 2^7 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot x^2 + 2^9 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 23 \cdot 71 \cdot x + 2^{14} \cdot 3^4 \cdot 5^4 \cdot 7^4 \cdot 11^5$$

via the change of variables $x = 16x_1 - 574928, y = 64y_1$ gives the minimal model E'

$$y_1^2 = x_1^3 + x_1^2 + 483312317x_1 + 14539257649013$$

Again one may easily check that the point $P = (-17428, -907137)$ is in $E'(\mathbb{Q})$. It has canonical height $\widehat{h}(P) = 5.106\ldots$ and thus it has infinite order. The CPS bound is computed to be $B = 10.774\ldots$. As before this allows one to show that P is a generator of $E'(\mathbb{Q})$. The point P corresponds to the point $(-853776, 58056768)$, a generator of $E(\mathbb{Q})$. Finally, using this basis in the computation of integral points in Sage yields that the only integral points are $(-853776, \pm 58056768)$ but x cannot be -853776, which is negative, and hence not of the required form. $\square$

6. Appendix: Equivariant Localization

For the sake of completeness we describe the well-known computation of the integrals

$$\int_{S^{[n]}} \delta^k \mathrm{c}_\mu(S^{[n]})$$

for $S = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ and $\delta = \det \mathcal{O}^{[n]}$ by toric localization.

First consider $S = \mathbb{A}^2$, which has an action by $G = \mathbb{G}_m^2$ via $(x, y) \mapsto (\alpha x, \beta y)$ where α, β are the characters obtained by projecting to each factor. The only fixed point is the origin $(0, 0)$. G also acts on $(\mathbb{A}^2)^{[n]}$; fixed points are length n subschemes Z fixed by G. Thus, they must be supported on a fixed point (i.e. the origin), and the ideal $I_Z \subset A = \mathbb{C}[x, y]$ must be generated by monomials. I_Z is determined by the monomials $x^a y^b$ left out of the ideal, which form a Young tableau with n boxes. Given such a Young tableau in the upper right quadrant, let $(i, b_i - 1)$ for $0 \le i \le n-1$ be the extremal boxes, so b_i is the height of the ith column. A partition μ of n uniquely determines a Young tableau by arranging μ_i columns of height i in descending order.

For a space X with an action by G with isolated fixed points, we can compute integrals over X by restricting to the fixed point locus using Bott localization:

$$\int_X \varphi = \sum_{p \in X^G} \int \frac{i_p^* \varphi}{\mathrm{c}_{\mathrm{top}}(T_p X)}$$

where $\varphi \in H_G^*(X)$, $i_p^* : H_G^*(X) \to H_G^*(X^G) \cong H^*(X^G) \otimes H_G^*([\mathrm{pt}])$ is the pull-back to a fixed point $p \in X^G$. The Chern class is the equivariant Chern class of the G representation $T_p X$.

For example, consider $S = \mathbb{A}^2$ again. For a partition μ representing a fixed point p_μ of $X = (\mathbb{A}^2)^{[n]}$, the Chern polynomial is [ES87, Lemma 3.2]

$$\begin{aligned} C(\mu; \alpha, \beta) &:= \sum_i t^i \mathrm{c}_{2n-i}(T_{p_\mu} X) \\ &= \prod_{1 \le i \le j \le n} \prod_{s=b_j}^{b_{j-1}-1} (t + (i - j - 1)\alpha + (b_{i-1} - s - 1)\beta)(t + (j - i)\alpha + (s - b_{i-1})\beta) \end{aligned}$$

$\mathcal{O}^{[n]}$ restricted to a point of $\mathbb{A}^{[n]}$ corresponding to a subscheme Z is canonically $\mathcal{O}_Z$, so setting $f = \mathrm{c}_1(\mathcal{O}^{[n]})$,

$$Z(\mu;\alpha,\beta) := i_{p_\mu}^* f = \sum_{i=0}^{n} \sum_{j=0}^{b_i-1} (i\alpha + j\beta)$$

6.1. **The case $S = \mathbb{P}^2$.** Let $\mathbb{G}_m^2$ act on $[x,y,z]$ via $[\alpha x, \beta y, z]$. There are three fixed points $p_0 = [0,0,1], p_1 = [0,1,0]$, $p_2 = [1,0,0]$, and a length n subscheme Z of $\mathbb{P}^2$ will consist of a length n_i subscheme Z_i at p_i with $\sum n_i = n$. The tangent space at such a point is canonically

$$T_Z(\mathbb{P}^2)^{[n]} = \bigoplus_i T_{Z_i}(\mathbb{P}^2)^{[n_i]}$$

Note that at any point $[Z] \in (\mathbb{P}^2)^{[n]}$ corresponding to a subscheme Z supported at p_i, there is a $\mathbb{G}_m^2$-stable Zariski neighborhood isomorphic to $\mathbb{A}^{[n]}$ with torus action via $(\alpha x, \beta y)$, $(\alpha\beta^{-1}x, \beta^{-1}y)$, $(\beta\alpha^{-1}x, \alpha^{-1}y)$ for $i = 0,1,2$ respectively. A 3-vector partition $\underline{\mu}$ of n will be three partitions (μ_1,μ_2,μ_3) such that $|\mu_1| + |\mu_2| + |\mu_3| = n$; 3-vector partitions of n classify fixed points $p_{\underline{\mu}}$ of $X = (\mathbb{P}^2)^{[n]}$. By the above, the tangent space at $p_{\underline{\mu}}$ has Chern polynomial

$$\sum t^i C_{2n-i}(\underline{\mu};\alpha,\beta) = C(\mu_1;\alpha,\beta)C(\mu_2;\alpha-\beta,-\beta)C(\mu_3;\beta-\alpha,-\alpha)$$

Define $C_i(\underline{\mu};\alpha,\beta) = \mathrm{c}_i(T_{p_{\underline{\mu}}}X)$. Also,

$$Z(\underline{\mu};\alpha,\beta) := i_{p_{\underline{\mu}}}^* f = Z(\mu_1;\alpha,\beta) + Z(\mu_2;\alpha-\beta,-\beta) + Z(\mu_3;\beta-\alpha,-\alpha)$$

The final answer is then, for $X = (\mathbb{P}^2)^{[n]}$

$$\int_X f^{2n-\sum_i k_i} \prod_i \mathrm{c}_{k_i}(TX) = \sum_{\underline{\mu},|\underline{\mu}|=n} \frac{Z(\underline{\mu};\alpha,\beta)^{2n-\sum_i k_i} \prod_i C_{k_i}(\underline{\mu};\alpha,\beta)}{C_{2n}(\underline{\mu};\alpha,\beta)}$$

6.2. **The case $S = \mathbb{P}^1 \times \mathbb{P}^1$.** Let $\mathbb{G}_m^2$ act on $S = \mathbb{P}^1 \times \mathbb{P}^1$ via $[\alpha x_1, y_1] \times [\beta x_2, y_2]$. The fixed points are classified by 4-vector partitions $\underline{\mu}$. Now we have

$$\sum t^i C'_{2n-i}(\underline{\mu};\alpha,\beta) = C(\mu_1;\alpha,\beta)C(\mu_2;-\alpha,\beta)C(\mu_3;\alpha,-\beta)C(\mu_4;-\alpha,-\beta)$$

Also,

$$Z'(\underline{\mu};\alpha,\beta) := i_{p_{\underline{\mu}}}^* f = Z(\mu_1;\alpha,\beta) + Z(\mu_2;-\alpha,\beta) + Z(\mu_3;\alpha,-\beta) + Z(\mu_4;-\alpha,-\beta)$$

The final answer is then once again

$$\int_{(\mathbb{P}^1\times\mathbb{P}^1)^{[n]}} f^{2n-\sum_i k_i} \prod_i \mathrm{c}_{k_i}(TX) = \sum_{\underline{\mu},|\underline{\mu}|=n} \frac{Z'(\underline{\mu};\alpha,\beta)^{2n-\sum_i k_i} \prod_i C'_{k_i}(\underline{\mu};\alpha,\beta)}{C'_{2n}(\underline{\mu};\alpha,\beta)}$$

6.3. **Universal Polynomials.** Let Φ be the universal genus from Section 3.1. We have

$$\sum_{n\geq 0} z^n \int_{S^{[n]}} \exp\det(\mathcal{O}^{[n]})\Phi(S^{[n]}) = \mathbf{A}(z)^{\mathrm{c}_1(S)^2}\mathbf{B}(z)^{\mathrm{c}_2(S)}$$

We have computed explicitly in SAGE the power series $\mathbf{A}$ and $\mathbf{B}$ for vanishing odd Chern classes up to degree 20, and the result can be found on the authors' webpages. For illustration, we include the formula up to degree 2:

$$\Phi = 1 + \mathrm{c}_2(a_1^2 - 2a_2) + \mathrm{c}_2^2(a_2^2 - 2a_1a_3 + 2a_4) + \mathrm{c}_4(a_1^4 - 4a_1^2a_2 + 2a_2^2 + 4a_1a_3 - 4a_4) + \cdots$$

By localization, we compute:

$$\begin{aligned}\mathbf{A}(z) &= 1 + a_2 z \\ &+ z^2\left(-a_1^3 + 3a_1^2a_2 + \frac{1}{4}a_1^2 + a_1a_2 - \frac{9}{2}a_2^2 + a_1a_3 + \frac{1}{6}a_1 - \frac{3}{2}a_2 + 3a_3 - 10a_4 - \frac{1}{48}\right) + O(z^3) \\ \mathbf{B}(z) &= 1 + z\left(a_1^2 - 2a_2\right) \\ &+ z^2\left(2a_1^4 - 8a_1^2a_2 - \frac{5}{4}a_1^2 + \frac{31}{2}a_2^2 - 15a_1a_3 + \frac{5}{2}a_2 + 15a_4 + \frac{1}{48}\right) + O(z^3)\end{aligned}$$

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