O-MINIMAL GAGA AND A CONJECTURE OF GRIFFITHS

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ABSTRACT. We prove a conjecture of Griffiths on the quasi-projectivity of images of period maps using algebraization results arising from o-minimal geometry. Specifically, we first develop a theory of analytic spaces and coherent sheaves that are definable with respect to a given o-minimal structure, and prove a GAGA-type theorem algebraizing definable coherent sheaves on complex algebraic spaces. We then combine this with algebraization theorems of Artin to show that proper definable images of complex algebraic spaces are algebraic. Applying this to period maps, we conclude that the images of period maps are quasi-projective and that the restriction of the Griffiths bundle is ample.

1. INTRODUCTION

Let X be a smooth algebraic variety of finite type over $\mathbb{C}$ supporting a pure polarized integral variation of Hodge structures $(V_{\mathbb{Z}}, F^\bullet, Q)$. Let Ω be the associated pure polarized period domain with generic Mumford–Tate group $\mathbf{G}$, and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ an arithmetic lattice containing the image of the monodromy representation of $V_{\mathbb{Z}}$. There is an associated period map $\varphi : X^{\mathrm{an}} \to \Gamma\backslash\Omega$, where X^{an} is the analytification of X, that is, $X(\mathbb{C})$ endowed with its natural structure as a complex analytic manifold. In general, for X a reduced separated algebraic space of finite type over $\mathbb{C}$, we refer to any holomorphic locally liftable map $\varphi : X^{\mathrm{an}} \to \Gamma\backslash\Omega$ satisfying Griffiths transversality as a period map.

Despite their transcendental nature, period maps are expected to be algebraic in the following sense. The Griffiths bundle $L := \bigotimes^i \det F^i$ exists universally on $\Gamma\backslash\Omega$ as a $\mathbb{Q}$-bundle and has natural positivity properties in Griffiths transverse directions. In [19, pg.259], Griffiths conjectured when φ is proper that the restriction of L to $\varphi(X^{\mathrm{an}})$ is in fact ample, realizing $\varphi(X^{\mathrm{an}})$ as the analytification of a quasi-projective variety. Note that $\Gamma\backslash\Omega$ itself rarely has an algebraic structure [10, 21].

Our main result is the following theorem, providing a solution to the conjecture:

Theorem 1.1. *Let X be a reduced separated algebraic space of finite type over $\mathbb{C}$ and $\varphi : X^{\mathrm{an}} \to \Gamma\backslash\Omega$ a period map as above. Then*

(1) φ factors (uniquely) as $\varphi = \iota \circ f^{\mathrm{an}}$ where $f : X \to Y$ is a dominant map of (reduced) finite-type algebraic spaces and $\iota : Y^{\mathrm{an}} \to \Gamma\backslash\Omega$ is a closed immersion of analytic spaces;

(2) the Griffiths $\mathbb{Q}$-bundle L restricted to Y is the analytification of an ample algebraic $\mathbb{Q}$-bundle, and in particular Y is a quasi-projective variety.

Note that if the period map φ is proper and if X and the Griffiths bundle on X are both defined over a subfield k of $\mathbb{C}$ (for example, if the variation comes from a smooth projective family defined over k), then it easily follows that the first map $g : X \to Y'$ in the Stein factorization $X \xrightarrow{g} Y' \xrightarrow{h} Y$ of f will also be defined over k.

As a sample application, we have the following immediate corollary[1]:

[1] We prove Theorem 1.1 and Corollary 1.2 also for the non-reduced case, subject to a natural admissibility condition arising from the o-minimal structure, which is satisfied for all variations coming

Corollary 1.2. *Let $\mathcal{M}$ be a reduced separated Deligne–Mumford stack of finite type over $\mathbb{C}$ admitting a quasi-finite period map. Then the coarse moduli space of $\mathcal{M}$ is quasi-projective.*

Corollary 1.2 for instance will apply to a reduced separated Deligne–Mumford moduli stack of smooth polarized varieties with an infinitesimal Torelli theorem. This provides an alternate approach to results of Viehweg [38] on the quasi-projectivity of (normalizations of) coarse moduli spaces of smooth polarized varieties X without assuming any positivity of K_X. In particular, Corollary 1.2 also applies to uniruled X provided deformations can be detected by Hodge theory.

The strategy of the proof of Theorem 1.1 hinges on algebraization results in o-minimal geometry. Briefly, an o-minimal structure specifies a class of "tame" subsets of $\mathbb{R}^n$ with strong finiteness properties. Such subsets are said to be definable with respect to the structure. The resulting geometric category of complex analytic varieties that are pieced together by definable charts (which we call definable analytic varieties, see section 2) on the one hand allows some of the local flexibility of the analytic category but on the other hand behaves globally like the algebraic category. An excellent example of this is the celebrated "definable Chow theorem" of Peterzil–Starchenko [34, Corollary 4.5], asserting that a closed complex analytic subvariety of a (not necessarily proper) complex algebraic variety which is definable in an o-minimal structure is in fact algebraic.

In [3], it is shown that $\Gamma\backslash\Omega$ is in this sense a definable analytic variety, and that period maps are definable with respect to this structure. To prove Theorem 1.1, we use Artin's theorems on the algebraization of formal modifications to inductively algebraize φ on strata. This requires one to look at nilpotent thickenings and thus deal with non-reduced spaces, even if one is only interested primarily in varieties. In fact, the naive generalization of Theorem 1.1 to non-reduced spaces is false, as we show in Example 5.3. One of the benefits of working in the definable analytic category is that it provides a natural admissibility condition to extend Theorem 1.1 to this setting, and we prove the more general statement in section 5.

To algebraize the nilpotent thickenings that arise when trying to apply Artin's theorem, we develop a theory of coherent sheaves in the definable analytic category, and a GAGA-type theorem for definable coherent sheaves:

Theorem 1.3. *Let X be a separated algebraic space of finite type over $\mathbb{C}$ and X^{def} the associated definable analytic space. The "definabilization" functor $\mathbf{Coh}(X) \to \mathbf{Coh}(X^{\mathrm{def}})$ is fully faithful, exact, and its essential image is closed under subobjects and quotients.*

It follows for example that definable coherent subsheaves of algebraic sheaves are algebraic. Note that X is *not* required to be proper over $\mathbb{C}$, but in contrast to Serre's classical GAGA theorem [35] (as well as most other GAGA-type theorems for proper algebraic spaces), it is *not* true that every definable coherent sheaf is algebraic (see Example 3.2).

The first part of Theorem 1.1 is then deduced from Theorem 1.3 and a more general theorem on the algebraicity of definable images of algebraic varieties:

Theorem 1.4. *Let X be a separated algebraic space of finite type over $\mathbb{C}$, $\mathcal{S}$ a definable analytic space, and $\varphi : X^{\mathrm{def}} \to \mathcal{S}$ a proper definable analytic map. Then $\varphi : X^{\mathrm{def}} \to \varphi(X^{\mathrm{def}})$ is (uniquely) the definabilization of a map of algebraic spaces.*

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from geometry. Moreover, a version of Theorem 1.1 for the image in the stack $[\Gamma\backslash\Omega]$ may be obtained by first passing to a level cover.

It is in general difficult to relate the metric positivity of the Hodge bundle to its ampleness on Y^{an} as the latter might be quite singular. One can, however, use it to show L is big and nef on a log resolution, and we finally show how to deduce the second statement in Theorem 1.1 from this and the algebraicity of Y.

Theorem 1.1 combined with the o-minimal algebraization results have a number of applications, and we describe a few in the final section including:

(1) A version of the Borel algebraicity theorem for period images (section 7.1).
(2) As a concrete example of Corollary 1.2, we deduce a general result about the quasi-projectivity of moduli spaces of complete intersections (section 7.2).
(3) A theorem showing that pure polarized integral variations of Hodge structures over dense Zariski open subsets of compact *Kähler manifolds* are pulled back from algebraic varieties (section 7.3).
(4) A version of the ampleness result in Theorem 1.1 for the *Hodge* bundle (section 7.4).

1.1. **Previous results.** Griffiths proved his conjecture in the case that the image $\varphi^{\mathrm{an}}(X^{\mathrm{an}})$ is compact [20, III.9.7]. Sommese [36] proved the conjecture in the case that the image has only isolated singularities, and later [37] proved a function field variant. In particular, he proved that the image of a period map admits a proper desingularization which is quasi-projective and such that the induced meromorphic map is rational. However, for example it does not follow from their works that period images admit a compactification by a compact analytic space.

There is ongoing work by Green–Griffiths–Laza–Robles [18] which attempts to produce a functorial compactification of period images, and goes on to deduce the quasi-projectivity from that. However, there is a gap in Proposition 3.4.7: essentially, the difficulty in this type of argument is to "glue" the period images from the associated graded variation of the mixed Hodge structures on the boundary to the period image inside. They are currently working to resolve the issue.

The subject of o-minimal sheaves and the development of a cohomology theory were treated in [14], and this was further developed in subsequent papers. Variants of the o-minimal Nullstellensatz and Weierstrass Preparation theorems were proven by Kaiser [24].

Kashiwara–Schapira [25] have constructed a subanalytic site as well as a theory of subanalytic sheaves which is in general different from our construction in section 2 for the subanalytic o-minimal structure $\mathbb{R}_{\mathrm{an}}$—see the beginning of section 2 for a more precise discussion. Petit [32] has defined a "tempered analytification" functor on smooth algebraic varieties and proven a conditional GAGA theorem reminiscent of Theorem 1.3 on the subanalytic sites of smooth algebraic varieties in the sense of Kashiwara–Schapira.

1.2. **Outline.** In section 2 we develop the theory of definable coherent sheaves and definable analytic spaces. We also define and prove some basic properties of the definabilization functor on algebraic spaces and the analytification functor on definable analytic spaces. In section 3 we prove Theorem 1.3 (see Theorem 3.1), and in section 4 we prove Theorem 1.4 (see Theorem 4.2). In sections 5 and 6 we prove stronger versions of the two parts of Theorem 1.1 allowing for non-reduced bases (see Theorem 5.4 and Theorem 6.2). In section 7 we deduce some applications, including Corollary 1.2 (see Corollary 7.3).

1.3. **Acknowledgements.** J.T. would like to thank Vivek Shende, Jonathan Pila, and Ryan Keast for useful conversations. B.B. would like to thank Valery Alexeev and Johan de Jong for useful conversations. Y.B. would like to thank Olivier Benoist, Patrick Brosnan, and Wushi Goldring for useful conversations. The authors would also like to thank Ariyan Javanpeykar for useful remarks, specifically regarding section 7.1. This paper, and in particular section 2 owes a lot to the works of Peterzil and Starchenko, who initiated the study of o-minimal complex geometry. B.B. was partially supported by NSF grant DMS-1702149.

1.4. **Notation.** All schemes and algebraic spaces are assumed to be separated and of finite type over $\mathbb{C}$, and all definable spaces, definable analytic spaces, and analytic spaces are assumed to be separable. When helpful (mostly in sections 3, 4, and 5), we will loosely adopt the convention that algebraic objects are denoted by roman letters, and (definable) analytic objects by script letters.

Throughout, we fix an o-minimal structure with respect to which we will use the word "definable". The reader unfamiliar with these notions may assume for concreteness the structure $\mathbb{R}_{\mathrm{alg}}$ for which the definable subsets of $\mathbb{R}^n$ are the real semi-algebraic subsets. For the applications to Hodge theory in sections 5 we restrict to the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$. For a general introduction to o-minimality, see [12], and [13] for a discussion of o-minimality in a similar language to this paper.

2. Definable analytic spaces

2.1. **Definitions.** We start by briefly recalling the notion of a definable space. A *definable space* is a topological space X with a finite atlas by definable charts with definable transition functions. A morphism of definable spaces is a continuous map which is definable on every chart. For any definable space X there is a natural *definable site* whose objects are the definable open subsets, and the admissible coverings are simply the finite coverings.[2] When necessary, we'll denote the definable site by $\underline{X}$. We sometimes abusively refer to sheaves on the definable site as sheaves on X. Given a continuous definable map $f : X \to Y$, there are in the usual way adjoint functors $f_* : \mathrm{Sh}(\underline{X}) \to \mathrm{Sh}(\underline{Y})$ and $f^{-1} : \mathrm{Sh}(\underline{Y}) \to \mathrm{Sh}(\underline{X})$ on the categories of sheaves.

For a definable open set $U \subset \mathbb{C}^n$ we let $\mathcal{O}(U)$ be the definable holomorphic functions on U, that is the maps $U \to \mathbb{C}$ that are both definable and holomorphic.

Lemma 2.1. *The functor $U' \to \mathcal{O}(U')$ is a sheaf on the definable site of U.*

Proof. Let U_i be a finite covering of U. If a function $f \in \mathcal{O}(U)$ vanishes on each U_i, it must be identically 0. Conversely, if f_i are definable holomorphic functions on U_i which agree on overlaps, they clearly glue to a single holomorphic function f on U. Since the U_i are a finite covering of U and each f_i is definable, it follows that f is also definable and hence $f \in \mathcal{O}(U)$ as required. □

Note that the stalks $\mathcal{O}_p := \lim_{p \in V} \mathcal{O}(V)$ are local rings.

[2] For a real analytic manifold M, Kashiwara–Schapira [25, §7] have introduced the *subanalytic site* M_{sa} of M whose objects consist of subanalytic open subsets of M and whose coverings satisfy a local finiteness condition: for any $U \in M_{\mathrm{sa}}$, any covering U_i of U, and any compact $K \subset M$, they require that the cover $K \cap U_i$ of $K \cap U$ has a finite refinement.

Our construction with respect to the subanalytic o-minimal structure $\mathbb{R}_{\mathrm{an}}$ (see e.g. [13]) in general yields a different category of sheaves than that of Kashiwara–Schapira since a noncompact real analytic manifold M does not have a canonical structure as a $\mathbb{R}_{\mathrm{an}}$-definable space. Indeed, the classical notion of subanalyticity of a subset $Z \subset \mathbb{R}^n$ (see e.g. [6]) is a *local* condition. $\mathbb{R}_{\mathrm{an}}$-definability of a subset $Z \subset \mathbb{R}^n$ is a stronger condition, which roughly says that Z is *globally* subanalytic up to a finite cover. The Kashiwara–Schapira site therefore contains all $\mathbb{R}_{\mathrm{an}}$-definable coverings of M for each $\mathbb{R}_{\mathrm{an}}$-definable space structure on M, and is consequently a much finer topology in general. A compact real analytic manifold M is uniquely a $\mathbb{R}_{\mathrm{an}}$-definable space, and in this case the Kashiwara–Schapira site and the $\mathbb{R}_{\mathrm{an}}$-definable site give equivalent categories of sheaves.

Definition 2.2. Given an open definable subset $U \subset \mathbb{C}^n$ and a finitely generated ideal I of $\mathcal{O}(U)$, $X = V(I)$ is naturally a definable space. There is a sheaf $\mathcal{O}/I\mathcal{O}$ on U which is supported on X. We set $\mathcal{O}_X$ be the restriction of $\mathcal{O}/I\mathcal{O}$ to X. We call the data $(X, \mathcal{O}_X)$ a *basic definable analytic space.*

Definition 2.3. If X is a definable space and $\mathcal{O}_X$ is a sheaf of rings on the definable site of X, then we call $(X, \mathcal{O}_X)$ a *definable analytic space* if it has a finite covering by basic definable analytic spaces. A morphism $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a morphism $\varphi : X \to Y$ of definable spaces and a homomorphism $\varphi^\sharp : \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ of sheaves of rings such that if $\varphi(x) = y$, then the induced map of local rings $\varphi^\sharp_{x,y} : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a local homomorphism.

If no confusion arises, we will abusively just refer to X as the definable analytic space. For X a definable analytic space, denote by $\mathrm{Mod}(\mathcal{O}_X)$ the category of $\mathcal{O}_X$-modules. Given a morphism $\varphi : X \to Y$ of definable analytic spaces, we naturally have a functor $\varphi_* : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_Y)$, and we define a functor $\varphi^* : \mathrm{Mod}(\mathcal{O}_Y) \to \mathrm{Mod}(\mathcal{O}_X)$ via

$$\varphi^* : F \mapsto \mathcal{O}_X \otimes_{\varphi^{-1}\mathcal{O}_Y} \varphi^{-1}F$$

where as usual we have used the adjoint map $\varphi^\sharp : \varphi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ to make $\mathcal{O}_X$ a $\varphi^{-1}\mathcal{O}_Y$-algebra.

Lemma 2.4. *Let X be a definable analytic space. Then elements of $\Gamma(X, \mathcal{O}_X)$ are in a natural bijection with maps of definable analytic spaces $X \to \mathbb{C}$.*

Proof. It is enough to consider X a basic definable analytic space $V(I)$ where I is a finitely generated ideal sheaf in a definable open set $U \subset \mathbb{C}^n$. Given $f \in \Gamma(X, \mathcal{O}_X)$, after passing to a definable cover f extends to a section $f_0 \in \Gamma(U, \mathcal{O}_U)$, and thus a map $\varphi(f_0) : U \to \mathbb{C}$ which restricts to a map $\varphi(f) : X \to \mathbb{C}$. Note that if we pick a different section $f_0 + i$, where $i \in \Gamma(U, I)$ then we get the same map. To see this, note that it is obvious that $\varphi(f_0), \varphi(f_0 + i)$ give the same map on points $X \to \mathbb{C}$. To check that the map on the sheaf of rings is the same, note that if g is a definable function on $\mathbb{C}$, then $\frac{g(f_0+i)-g(f_0)}{i}$ is a definable holomorphic function, and thus $g(f_0 + i) - g(f_0) \in I$.

For the other direction, given a map $f : X \to \mathbb{C}$ we get a map $f^\# : \Gamma(\mathbb{C}, \mathcal{O}_\mathbb{C}) \to f_*(\mathbb{C}, f_*\mathcal{O}_X)$ and we take the section $\psi(f) := f^\#(z)$. It is easy to check that ψ, φ are inverse to each other. ◻

For some of the proofs in the following subsections, we will need a notion of definable cell decomposition for definable spaces that is slightly different than the usual one for $\mathbb{R}^n$ (see for example [12, §3.2]).

Definition 2.5. Let X be a definable space.

1. By a *definable cell* in X we mean a definable subspace $D \subset X$ which is definably homeomorphic to $\mathbb{R}^k$ for some k.
2. A *definable cell predecomposition of* X is a pairwise disjoint finite set $\{D_i\}$ of definable cells of X such that $X = \bigsqcup_i D_i$.
3. A definable cell predecomposition $\{D_i\}$ of X is a *definable cell decomposition* if in addition the closure of any cell is a union of cells.

Proposition 2.6. *Let X be a definable space. A definable cell decomposition of X exists.*

Proof. The proof is obtained by combining the following two lemmas.

Lemma 2.7. *Let X be a definable space and $\{Y_j\}$ a finite set of definable subspaces. Then there is a definable cell predecomposition of X for which each Y_j is a union of cells.*

Proof. Let U_i be a definable atlas of X, with definable homeomorphisms $\varphi_i : U_i \to V_i$ onto definable $V_i \subset \mathbb{R}^n$. We first claim that X is a disjoint union of definable subspaces U_i' such that $U_i' \subset U_i$ for each i. Indeed, taking $U_1' = U_1 \backslash \bigcup_{i>1} U_i$, we reduce to proving the claim for $X \backslash U_1'$, and then we are done by induction. Now the lemma reduces to the case that $X = U_i'$ is definably homeomorphic to a subspace of $\mathbb{R}^n$ (with subspaces $\{Y_j \cap U_i'\}$), which follows from the usual definable cell decomposition. □

Lemma 2.8. *Let X be a definable space. Then any definable cell predecomposition can be refined to a definable cell decomposition.*

Proof. Let $\{D_i\}$ be a definable cell predecomposition. We proceed by induction on the maximal dimension n of the D_i. Divide the D_i into n-dimensional cells E_i and cells D_i' of dimension $< n$. Note that any closure $\overline{E}_i$ cannot intersect any other E_j (since X is separable), so ∂E_i is in the union of the D_i'. Applying the previous lemma to

$$X' := \bigcup D_i'$$

taking as subspaces all D_i' and all ∂E_i, we obtain a predecomposition of X' that refines $\{D_i'\}$ and for which each ∂E_i is a union of cells. Now applying the induction hypothesis, the claim follows. □

□

2.2. **Coherent sheaves.**

Definition 2.9. Given an $\mathcal{O}_X$-module M, we say that M is *locally finitely generated* if there exist a finite cover of X by definable open sets X_i, and surjections $\mathcal{O}_{X_i}^n \twoheadrightarrow M_{X_i}$ for some positive integer n on each of those open sets. We say that M is *coherent* if it is locally finitely generated, and given any definable open $V \subset X$ and any $\mathcal{O}_V$-module homomorphism $\varphi : \mathcal{O}_V^n \to M_V$, the kernel of φ is locally finitely generated.

Note that it easily follows that if M is a coherent $\mathcal{O}_X$-module and N is a locally finitely generated $\mathcal{O}_X$-submodule, then N is coherent. Moreover, the kernel of any homomorphism $M \to M'$ of coherent $\mathcal{O}_X$-modules is locally finitely generated and therefore coherent. The following is standard but we include the proof for completeness.

Lemma 2.10. *Let $0 \to M_1 \to M \to M_2 \to 0$ be an exact sequence of sheaves. If two of $\{M, M_1, M_2\}$ are coherent then so is the third.*

Proof.
(1) Assume M, M_1 are coherent. Since M is locally finitely generated, so is M_2. Let us now show that M_2 is coherent. Suppose $V \subset X$ is a definable open and $\varphi : \mathcal{O}_V^n \to M_{2|V}$ is any map. The map φ is determined by the image of a basis. Since M surjects onto M_2, by further restricting to a finite open cover we can assume that φ lifts to a map $\varphi' : \mathcal{O}_V^n \to M_{|V}$.

Now since M_1 is coherent we may choose a surjection $\psi : \mathcal{O}_V^m \to M_{1|V}$ by further restricting to a finite open cover. Consider $\psi \oplus \varphi' : \mathcal{O}_V^m \oplus \mathcal{O}_V^n \to M_{|V}$. Then the kernel of $\psi \oplus \varphi'$ is finitely generated since M is coherent, and surjects onto the kernel of φ. Thus the kernel of φ is finitely generated, and so M_2 is coherent.
(2) Assume M, M_2 are coherent. Then any map to M_1 is also a map to M, and thus has finitely generated kernel. Moreover, if $\varphi : \mathcal{O}_X^n \twoheadrightarrow M$, then the kernel of the induced map to M_2 is finitely generated since M_2 is coherent, and surjects to M_1.
(3) Assume M_1, M_2 are coherent. To see that M is locally finitely generated, we first restrict to a finite open covering so that one can choose surjections φ_i :

$\mathcal{O}_X^{n_1} \to M_i$. Now by further restricting, we may lift φ_2 to a map $\varphi_2' : \mathcal{O}_X^{n_2} \to M$. Now the map $\varphi_1 \oplus \varphi_2' : \mathcal{O}_X^{n_1+n_2} \to M$ is a surjection.

Finally, let $\varphi : \mathcal{O}_X^m \to M$ be any map. When continued to M_2, the kernel K of $\varphi_0 : \mathcal{O}_X^m \twoheadrightarrow M_2$ is locally finitely generated. The induced map from K to M_1 has locally finitely generated kernel, and this kernel is in fact $\ker \varphi$. This completes the proof.

□

2.3. **Definable Oka coherence.**

Lemma 2.11. *Let $U \subset \mathbb{C}^n$ be a definable open set, $P \in \mathcal{O}(U)[w]$ a monic polynomial in w with coefficients that are definable holomorphic functions on U. Let $X \subset U \times \mathbb{C}$ be a definable open set containing $V(P)$. Then given any definable holomorphic function f on X, one can uniquely write $f = QP + R$ for definable functions Q, R with Q, R holomorphic and $R \in \mathcal{O}(U)[w]$ of degree less than the degree of P.*

Proof. Note that the uniqueness is true even in the analytic category, so it suffices to show existence (i.e. that the unique holomorphic Q, R are definable). Let $V \subset U \times \mathbb{C}$ be the zero-locus of P. Let V_i be the irreducible analytic components of V and P_i be the minimal polynomial of w over V_i. Note that the V_i are definable sets and so each P_i is definable. Also, P must be a product of the P_i, and so by induction on $\deg P_i$ it suffices to prove the theorem for each P_i one at a time. We may thus assume that P is irreducible. Now let $U_1 \subset U$ be the dense open set where $P(w)$ has distinct roots. On U_1 the coefficients of R are the tuples $a_0, \ldots, a_{n-1}$ such that $w^n + \sum_{i=0}^{n-1} a_i w^i$ agrees with f on V. Thus, it follows that $R|_{U_1}$ is definable. Since U_1 is dense in U it follows that R is definable as well, since the graph of R is the closure of the graph of R_1. Hence Q is definable since $Q = \frac{f-R}{P}$, and the proof is complete. □

For the proof of the following theorem, we need the following definition:

Definition 2.12. Let $f : X \to Y$ be a map of definable spaces. We say that f is *quasi-finite* if it has finite fibers, and f is *finite* if it is quasi-finite and proper.

Theorem 2.13. *Given a definable open set $X \subset \mathbb{C}^n$, the definable structure sheaf $\mathcal{O}_X$ is a coherent sheaf.*

Proof. Let $(f_1, \ldots, f_m)$ be definable holomorphic functions on X. This corresponds to a map $\varphi : \mathcal{O}_X^m \to \mathcal{O}_X$, and we need to show that the relation sheaf $I(\vec{f}) := \ker \varphi$ is locally finitely generated. Let Y be the zero set of f_1. On the complement of Y, a basis for $I(\vec{f})$ is given by

$$e_1 - \left(\frac{f_j}{f_1}\right) e_j$$

where the e_i are the standard basis of $\mathcal{O}_X^m$. It remains to show that $I(\vec{f})$ is locally finitely generated on (a neighborhood containing) Y. Using [33, Theorem 2.14], there is a covering of X by finitely many definable open sets X_i such that for each X_i, there is a linear set of coordinates for which $Y \cap X_i$ is proper over its projection down to $\mathbb{C}^{n-1}$. Note that the only compact, definable analytic subsets of $\mathbb{C}$ are finite, so replacing Y with $Y \cap X_i$ we may assume without loss of generality that there is a linear projection $\pi : Y \to \mathbb{C}^{n-1}$ which is definably proper and finite over its image $U := \pi(Y) \subset \mathbb{C}^{n-1}$. It follows that U is a definable open set.

Now, let Y_i be the irreducible components of Y, and let $P_i(w) \in \mathcal{O}(U)[w]$ be the unique irreducible polynomials whose zero-locus is Y_i. Note that $P_i(w)$ is definable since it can be defined as $P_i(w, u) = \prod_{t \in Y_i \cap \pi^{-1}(u)} (w - t)$. By the analytic Weierstrass preparation theorem, there are positive integers k_i such that $\frac{f_1}{\prod_i P_i(w)^{k_i}}$ is nowhere vanishing, and thus must be a definable unit. Hence we may assume $f_1 = P(w) = \prod_i P_i(w)^{k_i}$.

Let $k = \deg P$. By Lemma 2.11, we can replace the f_j by elements of $\mathcal{O}(U)[w]$ of degree less than k and so we do this. Now, $I(\vec{f})(X)$ contains the relations $-f_j e_1 + f_1 e_j$ for all j. It follows again by Lemma 2.11 that the remaining relations are generated by relations of the form $\sum g_j e_j$ for $g_j \in \mathcal{O}(U)[w]$ of degree less than k. We will show that such relations are locally finitely generated over $\mathcal{O}(U)$. In fact, separating out the coefficients of all polynomials in w in the map $(r_1, \ldots, r_m) \to \sum f_i r_i$ we get a map $\psi : \mathcal{O}_U^{mk} \to \mathcal{O}_U^k$. By induction, $\mathcal{O}_U$ is coherent and so by Lemma 2.10 the kernel of ψ is locally finitely generated over $\mathcal{O}_U$. Thus (by possibly further restricting to a subcover) we obtain finitely many relations $\vec{R}_i \in \mathcal{O}(U)[w]^m$ which together with the relations $e_1 - \left(\frac{f_j}{f_1}\right) e_j$ generate $\ker \varphi$ on any definable open subset of Y of the form $Y \cap \pi^{-1}(U_1)$ for a definable open $U_1 \subset U$. We claim that in fact these relations generate everywhere.

Lemma 2.14. *Let $f : X \to Y$ be a continuous, finite map of separated definable* spaces*, and let X_i be a definable open cover over of X. Then there is a refinement W_i of X_i and a definable open cover Y_i of Y such that each $f^{-1}(Y_i)$ is a disjoint union of the W_i.*

Proof. Consider the boolean algebra generated by the $f(X_i)$, and by Proposition 2.6 refine that to a decomposition of Y as a union of cells C_i. Refining further, we may assume that $f^{-1}(C_i)$ is a disjoint union of copies of C_i mapping down bijectively. Now let the D_i be the components of the $f^{-1}(C_i)$. Note that the D_i refine the X_i.

Now choose a cell D of X and let $X(D)$ be the union of cells of X whose closure contains D; likewise for a cell C of Y define $Y(C)$. It is clear that each $X(D)$ is definable and open in X. Moreover, if $D \subset X_i$ then it is clear that $X(D) \subset X_i$. Likewise for $Y(C)$.

Now suppose C is cell in Y, and D, D' are distinct cells in X mapping to C. Since X is separated, it follows that $X(D)$ and $X(D')$ are disjoint.

We claim now that $f^{-1}(Y(C))$ is the disjoint union of $X(D)$ for cells D of X mapping to C. To see this, if is sufficient to know that if C' is a cell containing C in its closure, then every lift D' of C' contains in its closure *some* lift of C. This is immediate since the map f is proper.

Finally, for any cell D of X, each cell of $Y(f(D))$ lifts to a cell of $X(D)$ by construction, so $X(D)$ is proper over its image $Y(f(D))$. The claim is now proven taking the W_i to be the $Y(C_i)$. □

Now, to see that our set of relations generates everywhere, it is sufficient by Lemma 2.14 to restrict to definable open sets W such that $W \cap Y$ is proper over its image $U' \subset U$. But then $W \cap Y$ is an open component of $Y \cap \pi^{-1}(U')$ and any relation on $W \cap Y$ can be extended by 0 to a relation on $Y \cap \pi^{-1}(U')$. Since our relations generate on $Y \cap \pi^{-1}(U')$, the proof is complete. □

We deduce for later use the following corollary of Lemma 2.14

Corollary 2.15. *Let $f : X \to Y$ be a continuous finite map of definable spaces. Then f_* is exact on the categories of abelian sheaves.*

Proof. Let $A \to B \to C$ be an exact sequence of sheaves on X; we want to prove the exactness of $f_*A \to f_*B \to f_*C$. For a definable open U of Y, if a section s in $f_*B(U) = B(f^{-1}(U))$ is zero in $f_*C(U)$, then taking after taking an open definable cover of $f^{-1}(U)$, s is in the image of A. By Lemma 2.14 we refine our open definable cover by components of $f^{-1}(W_i)$, where W_i are an open cover of Y. It follows that for each i, $s(W_i)$ is in the image of $f_*A(W_i)$, completing the proof. □

Theorem 2.13 implies the same coherence statement in general:

Theorem 2.16 (Oka coherence)**.** *Given a definable analytic space X, the structure sheaf $\mathcal{O}_X$ is a coherent sheaf.*

Proof. It is sufficient to assume that X is a basic definable analytic space. So let $X = V(I)$ where $U \subset \mathbb{C}^n$ is definable open and $I \subset \mathcal{O}_U$ is a finitely generated subsheaf. Let $i : X \to U$ be the natural injection. Note that $F \to i_*F$ gives an equivalence of categories between $\mathcal{O}_X$-modules on X and $\mathcal{O}_U$-modules on U killed by I, with inverse $F \to i^{-1}F$. Now if $\varphi : \mathcal{O}_X^n \to \mathcal{O}_X$ is a map, we may consider the map $i_*\varphi : i_*O_X^n \to i_*\mathcal{O}_X$ which is a map of coherent $\mathcal{O}_U$-modules. Since $\mathcal{O}_U$ is coherent, we may form an exact sequence $\mathcal{O}_U^t \to i_*O_X^n \to i_*\mathcal{O}_X$. The first map is killed by I, so we get an exact sequence $i_*\mathcal{O}_X^t \to i_*O_X^n \to i_*\mathcal{O}_X$, and thus an exact sequence $\mathcal{O}_X^t \to \mathcal{O}_X^n \to \mathcal{O}_X$ as desired.

□

We have implicitly used in the previous theorem (for $X \subset \mathbb{C}^n$ open) the following immediate corollaries, which are true from generalities about coherent sheaves:

Corollary 2.17.

(1) *$\mathcal{O}_X^n$ is coherent for any n.*
(2) *Any locally finitely generated submodule of a coherent sheaf is coherent.*
(3) *Any locally finitely presented $\mathcal{O}_X$ module is coherent.*

2.4. **Étale descent.** In order to definabilize algebraic spaces[3] in the next section, we need to take quotients by (not necessarily proper) étale equivalence relations. For U a definable analytic space, we say a definable analytic subspace $R \subset U \times U$ is an equivalence relation if for any definable analytic space S, $\mathrm{Hom}(S, R) \subset \mathrm{Hom}(S, U) \times \mathrm{Hom}(S, U)$ is an equivalence relation. We say that a map is étale if it is open and locally an isomorphism onto its image, and an equivalence relation is étale if the projection maps are étale.

Proposition 2.18. *Let U be a definable analytic space. Given a closed étale definable equivalence relation $R \subset U \times U$, there exist finitely many definable open sets U_i of U such that $R \cap (U_i \times U_i) = \Delta_{U_i}$, and the U_i collectively contain each R-equivalence class at least once.*

Proof.

Step 1. By definable choice, we can find a definable subset T of U which has exactly one point for each R-representative class. Let us stratify T by submanifolds T_i. Now for each i let S_i be the set of all points equivalent to T_i but not actually in T_i. It is easy to see that S_i is also a submanifold. Now we will show how to further stratify such that T_i is disjoint from $\overline{S_i}$. To do this, note that $T_i \cap \overline{S_i}$ is of smaller dimension than T_i. Thus by successively iterating in this way we can obtain our desired stratification. By further stratifying, we can assume that the number of R-pre-images along T_i is constant, and that each T_i is a cell and is therefore simply connected.

Step 2. By the argument in Lemma 2.14 we may take V_i to be a definable open neighbourhood of T_i such that $R \cap (V_i \times U)$ consists of k étale sections over V_i - which we denote by R_0 - and another piece R' which does not intersect $T_i \times U$.

[3]Recall that we need to work with algebraic spaces, since Artin's algebraization theorem does not hold true for schemes.

Step 3. Pick a definable distance function $d(x,y)$ on $U \times U$, and pick a definable exhaustion function $E : U \to \mathbb{R}_{\geq 0}$. In other words, $E^{-1}([0,c])$ is compact for all $c \in \mathbb{R}$. For a set $S \subset U$ we write S^c to mean $S \cap E^{-1}([0,c])$.

Step 4. By definable choice we may let $h : \mathbb{R}^2_{\geq 0} \to (0,1)$ be a definable, positive ϵ such that $R' \cap B_{d,\epsilon}(T_i^c) \times B_{d,\epsilon}(T_i^{c'}) = \emptyset$. Consider the function

$$g(c) := \min_{c_1,c_2<c} \frac{h(c_1,c_2)}{2}.$$

We let $f(c)$ be a definable positive, continuous, decreasing function strictly smaller than $g(c)$. Note that $h(c,c') > \min(f(c), f(c'))$.

Step 5. Define $d'(u,T_i) := \min_{c,t\in T_i^c} d(u,t)f(c)^{-1}$. Define W_i to consist of all points $u \in V_i$ such that $d'(u,T_i) < \min_{u'\in R(u)\setminus u} d'(u',T_i)$. We claim that W_i contains an open neighbourhood around T_i. Let $t \in T_i$. For $\epsilon > 0$, consider the ball $B_{d',\epsilon}(t)$. It is clear that for sufficiently small ϵ, d' is smaller on this ball then on R_0, and d' is smaller then $1/2$. Suppose that $u \in B_{d',\epsilon}(t), u' \in R'$ and $d(u',t') \leq f(c)$ for $t' \in T_i^c$. It follows that the point $(u,u') \in R' \cap B_{d,\epsilon}(T_i^c) \times B_{d,\epsilon}(T_i^{c'})$ for $\epsilon = \min(f(c), f(c')) < h(c,c')$. This is a contradiction. Now set $U_i \subset W_i$ to be the maximal open subset (which is a definable condition). This completes the proof.

□

For a definable equivalence relation $R \subset U \times U$, we say $U \to X$ is a quotient if it represents the sheaf U/R with respect to the definable topology. Concretely, this means that a map $S \to X$ is given by taking a definable cover S_i of S and giving maps $S_i \to U$ that agree on overlaps up to the equivalence relation. A quotient is unique up to unique isomorphism provided it exists.

Corollary 2.19. *Quotients by closed étale equivalence relations exist in the category of definable analytic spaces.*

Proof. The quotient can be glued together from the cover provided from Proposition 2.18. □

Corollary 2.20. *Let X, Y be definable analytic spaces and $f : X \to Y$ an étale morphism. Then there is a definable open cover X_i of X such that the restrictions $f_j : X_j \to Y$ are open immersions.*

Proof. Apply the proposition to the equivalence relation $X \times_Y X \subset X \times X$. □

For X a definable analytic space, let $\underline{X}$ be its definable site. Let $\underline{X}^{\text{ét}}$ be the site whose objects are definable analytic spaces with an étale morphism to X and whose covers are surjective such maps. The obvious inclusion $i : \underline{X} \to \underline{X}^{\text{ét}}$ yields a morphism of sites

$$f : \underline{X}^{\text{ét}} \to \underline{X}.$$

Recall this means i is continuous (sends covers to covers and respects fiber products with covers) and that the pullback on sheaves f^{-1} (which is just sheafification in the definable étale topology) is exact, both of which are immediate. The corollary implies the natural presheaf $\mathcal{O}_{X^{\text{ét}}} : U \to \mathcal{O}_U(U)$ on $\underline{X}^{\text{ét}}$ is a sheaf.

Corollary 2.21. *Let X be a definable analytic space. Then $f^{-1} : \mathrm{Sh}(\underline{X}) \to \mathrm{Sh}(\underline{X}^{\text{ét}})$ is an equivalence of the categories of sheaves. Moreover, $f^{-1}\mathcal{O}_X \to \mathcal{O}_{X^{\text{ét}}}$ is an isomorphism.*

Proof. By the above corollary, every definable étale cover is refined by a definable open cover. □

2.5. **Definabilization.** If X is an affine scheme presented as $\operatorname{Spec}\mathbb{C}[x_1,\dots,x_n]/I$ we define the definabilization X^{def} to be the definable analytic subspace of $\mathbb{C}^n$ given by the coherent ideal sheaf $I\mathcal{O}^{\mathrm{def}}_{\mathbb{C}^n}$. It is easy to see this yields a functor from affine schemes to definable analytic spaces which is functorial and maps open covers to open covers, and thereby extends uniquely to a functor from schemes to definable analytic spaces.

For X a scheme, the definabilization functor yields a morphism of ringed sites

$$g : (\underline{(X^{\mathrm{def}})}, \mathcal{O}_{X^{\mathrm{def}}}) \to (\underline{X}, \mathcal{O}_X)$$

as there is a natural map $g^{-1}\mathcal{O}_X \to \mathcal{O}_{X^{\mathrm{def}}}$. Let $\mathbf{Coh}(X)$ be the category of coherent sheaves on X, and $\mathbf{Coh}(X^{\mathrm{def}})$ the category of definable coherent sheaves on X^{def}. We then define a definabilization functor

$$\mathrm{Def} : \mathbf{Coh}(X) \to \mathbf{Coh}(X^{\mathrm{def}}) : F \mapsto F^{\mathrm{def}} := \mathcal{O}_{X^{\mathrm{def}}} \otimes_{g^{-1}\mathcal{O}_X} g^{-1}F.$$

Evidently there is a natural isomorphism $\mathcal{O}_X^{\mathrm{def}} \cong \mathcal{O}_{X^{\mathrm{def}}}$.

Suppose now that X is an algebraic space, presented as the quotient of U by a closed equivalence relation $R \subset U \times U$ where R, U are schemes and the projection maps are étale. We obtain a definable étale equivalence relation $R^{\mathrm{def}} \subset U^{\mathrm{def}} \times U^{\mathrm{def}}$, and by Corollary 2.19 we can define the definabilization X^{def} of X to be the categorical quotient. As morphisms of algebraic spaces are étale locally morphisms of schemes, this provides us with a definabilization functor from algebraic spaces to definable analytic spaces.

To extend the functor Def to X, there are two ways to proceed. There is a definabilization functor from the étale site of X to the étale site of X^{def}, and Corollary 2.21 allows us to define a functor $\mathrm{Def} : \mathbf{Coh}(X) \to \mathbf{Coh}(X^{\mathrm{def}})$. Alternatively, given the above presentation, let π_i and π_{ij} be the obvious projections $R \to U$, $R \times_U R \to R$, respectively. $\mathbf{Coh}(X)$ is then equivalent to the category of descent data: pairs (F, φ) where $F \in \mathbf{Coh}(U)$ and $\varphi : \pi_1^*F \to \pi_2^*F$ is an isomorphism such that on $R \times_U R$ we have $\pi_{13}^*\varphi = \pi_{23}^*\varphi \circ \pi_{12}^*\varphi$. $\mathbf{Coh}(X^{\mathrm{def}})$ has the analogous description, and the definabilization functor is simply $\mathrm{Def} : (F, \varphi) \mapsto (F^{\mathrm{def}}, \varphi^{\mathrm{def}})$, which can be easily seen to be independent of the choice of presentation.

2.6. **Analytification.** Likewise, there is a natural analytification functor from definable analytic spaces to analytic spaces which we denote by $X \mapsto X^{\mathrm{an}}$, as well as an analytification functor $\mathrm{An} : \mathbf{Coh}(X) \to \mathbf{Coh}(X^{\mathrm{an}})$ on the corresponding categories of coherent sheaves, with a natural identification $\mathcal{O}_X^{\mathrm{an}} \cong \mathcal{O}_{X^{\mathrm{an}}}$.

Theorem 2.22. *Let X be a definable analytic space and* $\mathrm{An} : \mathbf{Coh}(X) \to \mathbf{Coh}(X^{\mathrm{an}})$ *the analytification functor. Then*

(1) An *is faithful;*
(2) An *is exact.*

Proof of Theorem 2.22(1). We need to show that if we have $E \xrightarrow{f} F$ in $\mathbf{Coh}(X)$ such that $f^{\mathrm{an}} = 0$, then $f = 0$. By considering the image, it is enough to show that if for $F \in \mathbf{Coh}(X)$ we have $F^{\mathrm{an}} = 0$, then $F = 0$. The statement is local, so we may assume F has a presentation

$$\mathcal{O}_X^m \xrightarrow{g} \mathcal{O}_X^n \to F \to 0$$

and we reduce to the following lemma.

Lemma 2.23. *If g^{an} is surjective then g is.*

Proof. We can express g as an $m \times n$ matrix M consisting of elements of $\mathcal{O}_X(X)$. Since g^{an} is surjective at every point of X some $n \times n$ minor of M is invertible, and so its inverse is definable as it is a rational function of the entries of g. It follows that g is surjective. □

□

For the second part of Theorem 2.22, we first need some preliminary observations.

Lemma 2.24. *For X a definable analytic space, and $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a Noetherian ring.*

Proof. Without loss of generality $X \subset U \subset \mathbb{C}^k$ is a basic definable analytic space $V(I)$ for I a finitely generated ideal. Then $\mathcal{O}_{X,p}$ is a quotient of $\mathcal{O}_{\mathbb{C}^k,p}$. Thus it is sufficient to prove $\mathcal{O}_{\mathbb{C}^k,p}$ is Noetherian.

We proceed by induction on k. Suppose $0 \neq f \in \mathcal{O}_{\mathbb{C}^k,p}$. By [33, Theorem 2.14] we can change coordinates such that f is a unit times a Weierstrass polynomial $P(w) \in \mathcal{O}_{\mathbb{C}^{k-1},p}[w]$. Thus $\mathcal{O}_{\mathbb{C}^k,p}/(f)$ is finite over $\mathcal{O}_{\mathbb{C}^{k-1},p}$. The theorem thus follows by induction, since a finite extension of a noetherian ring is noetherian. □

Lemma 2.25. *For X a definable analytic space, $p \in X$, the completions of $\mathcal{O}_{X,p}$ and $\mathcal{O}^{\mathrm{an}}_{X,p}$ are canonically isomorphic.*

Proof. For X an open set in $\mathbb{C}^n$ the claim is clear since both completions are canonically the formal power series ring R_n in n variables. By the Artin-Rees lemma, it follows that tensoring with $\mathcal{O}^{\mathrm{an}}_{\mathbb{C}^n,p}$ over $\mathcal{O}_{\mathbb{C}^n,p}$ is exact for finitely generated modules.

Now suppose $X \subset U$ is a basic definable analytic space cut out by an ideal sheaf I. By the above $I^{\mathrm{an}}_p := I_p \otimes_{\mathcal{O}_{U,p}} \mathcal{O}^{\mathrm{an}}_{U,p}$ is a subsheaf of $\mathcal{O}^{\mathrm{an}}_{U,p}$, and we have the isomorphisms

$$\mathcal{O}_{X,p} \cong \mathcal{O}_{U,p}/I_p, \text{ and } \mathcal{O}^{\mathrm{an}}_{X,p} \cong \mathcal{O}^{\mathrm{an}}_{U,p}/I^{\mathrm{an}}_p$$

It follows that the completions of $\mathcal{O}_{X,p}$ and $\mathcal{O}^{\mathrm{an}}_{X,p}$ are both isomorphic to $R_n/(I_p \otimes_{\mathcal{O}_{U,p}} R_n)$. □

Proof of Theorem 2.22(2). Sheafification in the analytic topology is exact and tensor products are always right exact, so it is sufficient to prove left-exactness of the tensor product. Suppose that $0 \to E \to F$ is an exact sequence of definable coherent sheaves. Then we get an injection of stalks $0 \to E_p \to F_p$ for $p \in X$. Now to show that E^{an} injects into F^{an} it is sufficient to prove that E^{an}_p injects into F^{an}_p. Note that $E^{\mathrm{an}}_p \cong E_p \otimes_{\mathcal{O}_{X,p}} \mathcal{O}^{\mathrm{an}}_{X,p}$. Since the completions of $\mathcal{O}_{X,p}$ and $\mathcal{O}^{\mathrm{an}}_{X,p}$ are canonically isomorphic by Lemma 2.25, and the rings are both Noetherian by Lemma 2.24, it follows that tensor product is exact on finitely generated modules. The claim therefore follows. □

Corollary 2.26. *Let X be a definable analytic space and $E \in$ **Coh**(X). For any section $s \in E(X)$, $s = 0$ if and only if $(s^{\mathrm{an}})_x = 0$ in $(E^{\mathrm{an}})_x$ for all $x \in X$.*

Proof. Consider the map $\varphi : \mathcal{O}_X \to E$ with $\varphi(1) = s$. Now $s = 0$ if and only if φ defines the 0 map, and by Theorem 2.22 this happens iff φ^{an} is the 0 map, which can be checked on stalks. □

Corollary 2.27. *Given a subsheaf $E \subset F$ and a section $s \in F(X)$, then $s \in E(X)$ iff $(s^{\mathrm{an}})_x \in (E^{\mathrm{an}})_x$ for all $x \in X$.*

2.7. **Reduced spaces.**

Lemma 2.28. *Let $U \subset \mathbb{C}^n$ be an open definable subset, and $\mathcal{Y} \subset U^{\mathrm{an}}$ a closed definable analytic set. Then the ideal sheaf $I \subset \mathcal{O}_U$ of functions vanishing on $\mathcal{Y}$ is coherent.*

Proof. By [33, Theorem 11.1], there is a finitely generated ideal sheaf $J \subset I \subset \mathcal{O}_U$ which agrees with I on stalks. We claim that $I = J$. Suppose $t \in I(U)$. Consider the finitely generated sheaf J' generated by J and t. Then J and J' have the same stalks, and are both coherent, and therefore $J = J'$ be Theorem 2.22. Thus, $t \in J(U)$ and so it follows that $I = J$. □

For X a definable analytic space and $\mathcal{Y} \subset X^{\mathrm{an}}$ a closed definable analytic set, we may locally endow $\mathcal{Y}$ with the reduced induced structure given by the ideal in the lemma, and thereby obtain a closed definable analytic subspace $Y \subset X$. Moreover, this structure is functorial in the sense that if $X \subset X'$ is a closed immersion of definable analytic spaces, and $\mathcal{Y} \subset X$ is a definable analytic set, then the reduced induced structures inherited from X and from X' are the same.

For X a definable analytic space, taking the underlying definable space we have its reduced space $X^{\mathrm{red}} \subset X$. We say that X is reduced if this embedding is an isomorphism. Note that it is clear that being reduced is equivalent to all the stalks $\mathcal{O}_{X,p}$ being reduced local rings.

2.8. Noetherian induction and the Nullstellensatz.

Proposition 2.29 (Definable Noetherian induction)**.** *Let X be a definable analytic space and F a coherent sheaf on X. Any increasing chain of coherent subsheaves of F must stabilize.*

Proof. It is enough to prove the statement on a definable cover. As F is locally a quotient of $\mathcal{O}_X^n$, and by pulling back our chain we may assume $F = \mathcal{O}_X^n$. The statement for $\mathcal{O}_X^n$ clearly follows from the statement for $\mathcal{O}_X$ so we may assume $F = \mathcal{O}_X$. We may take X to be a basic definable analytic space, and then as $\mathcal{O}_X$ is a quotient of $\mathcal{O}_{\mathbb{C}^n}$ we assume $X \subset \mathbb{C}^n$ is an open definable set.

We now induct on n to show the claim for $\mathcal{O}_X$ for $X \subset \mathbb{C}^n$ open. Our chain of definable coherent subsheaves corresponds to a chain of ideal sheaves I_j. We may assume after passing to a further cover that all of the I_j contain a function $f \in \mathcal{O}_X(X)$. As in the proof of Theoren 2.13, we may assume we have $Y \subset X \subset U \times \mathbb{C}$ where Y is the zero-locus of f, $P \in \mathcal{O}(U)[w]$ is a Weierstrass polynomial vanishing on Y, Y is definably proper over U, and the I_j contain P. Let $Q_j = I_j/P\mathcal{O}_X$ and $\pi : X \to U$ the projection. Note that the Q_j are coherent sheaves on Y and it is sufficient to show that that the Q_j stabilize.

Lemma 2.30. *With the above notation, let $\pi : Y \to U$ be the projection map. Then the pushforward map π_* takes coherent sheaves to coherent sheaves.*

Proof. By Lemma 2.11 we know that $\pi_*\mathcal{O}_Y \cong \mathcal{O}_U^{\deg P}$. Now let Q be a coherent sheaf. This means that Q has a presentation on a definable open cover, and by Corollary 2.15 this yields a presentation of π_*Q. □

By induction, the sequence π_*Q_j stabilizes. The theorem will thus follow if we show that $\pi_*Q_j = \pi_*Q_{j+1}$ implies that $Q_j = Q_{j+1}$. By Corollary 2.15 the pushforward π_* is exact, and thus it suffices to show that for a coherent sheaf Q, $\pi_*Q = 0$ implies that $Q = 0$. It is easy to see that $(\pi_*Q)_u = \oplus_{\pi(y)=u} Q_y$ and thus if $\pi_*Q = 0$ it follows that all stalks of Q are 0. The claim now follows from Theorem 2.22. □

Let X be a definable analytic space. For any definable coherent sheaf F, we define the the support $\mathrm{Supp}(F)$ as a definable analytic subspace as follows: if $\mathcal{O}_X^n \xrightarrow{g} \mathcal{O}_X^m \to F$ is a local presentation, we take $\mathrm{Supp}(F)$ to be the subspace cut out by the minors of g.

Lemma 2.31.

(1) The underlying definable analytic set of $\mathrm{Supp}(F)$ *is the set of $p \in X$ for which $F_p \neq 0$.*
(2) $\mathrm{Supp}(F)^{\mathrm{an}} = \mathrm{Supp}(F^{\mathrm{an}})$*.*
(3) The ideal of $\mathrm{Supp}(F)$ *in X is the ideal of functions $f \in \mathcal{O}_X$ such that $fF = 0$.*

Proof. For the first claim, $F_p \neq 0$ if and only if $g(p)$ is not surjective if and only if all minors of $g(p)$ are zero. For the second, a presentation analytifies to a presentation, by Theorem 2.22. For the third, let I be the ideal sheaf of $\mathrm{Supp}(F)$ in X. It follows that I^{an} is the ideal of the analytification of $\mathrm{Supp}(F)$ by Theorem 2.22. Now apply Corollary 2.27. □

Corollary 2.32. *Let X be a definable analytic space.*

(1) Any decreasing chain of closed definable analytic subspaces stabilizes.
(2) Any decreasing chain of closed definable analytic sets stabilizes.

Proof. For (1), consider the corresponding chain of ideals. This also handles (2), by endowing the subsets with the reduced induced structure. Note that by the lemma a definable analytic set $\mathcal{Y}$ may be recovered by the ideal sheaf I_Y defining the subspace Y with the reduced induced structure as the underlying set of $\mathrm{Supp}(\mathcal{O}_X/I_Y)$. □

Corollary 2.33 (Nullstellensatz)**.** *Let X be a definable analytic space. For some n, we have $I^n_{X^{\mathrm{red}}} = 0$.*

Proof. Let X_k be the definable analytic subspace given by the ideal $I^k_{X^{\mathrm{red}}}$. By the previous lemma, for any inclusion of definable coherent sheaves $E \subset E'$ we have $\mathrm{Supp}(E) \subset \mathrm{Supp}(E')$. Thus, $\mathrm{Supp}(I^k_{X^{\mathrm{red}}})$ gives a decreasing chain of definable analytic subspaces, which must eventually not contain any given point. Therefore, by Corollary 2.32 $\mathrm{Supp}(I_{X^{\mathrm{red}}})$ is eventually empty, and by the lemma $I^k_{X^{\mathrm{red}}} = 0$. □

Corollary 2.34. *Let X be a definable analytic space and $Z \subset X$ a definable analytic subspace. Then for some n, $I^n_{Z^{\mathrm{red}}} \subset I_Z$.*

2.9. **Descending analytic maps.** The purpose of this section is to prove a descent statement. In preparation, we need the following 2 lemmas:

Lemma 2.35. *Let Y be a definable analytic space and $Z \to Y \times \mathbb{C}$ be a closed definable analytic subspace, finite over Y. The projection map $\pi : Z \to Y$ is such that π_* maps coherent sheaves to coherent sheaves, and commutes with analytification.*

Proof. Let w be the last coordinate in $Y \times \mathbb{C}$. We claim that, after passing to a definable cover w satisfies a monic polynomial equation over $\mathcal{O}_Y(Y)$. By Corollary 2.33 we may assume that Z is irreducible and reduced, and thus also that Y is reduced. Replacing Y be the image of Z in Y, we may assume that Z maps surjectively onto Y, and thus that the number of pre-images is constant. Then satisfies the polynomial $P(x) := \prod_{(y,z)\in Z}(x - z) \in \mathcal{O}_Y[x]$.

Now let W be the analytic subspace cut out by P. We claim that $\psi_*\mathcal{O}_W$ is free over $\mathcal{O}_Y$. When Y is a domain in $\mathbb{C}^n$, this follows from Lemma 2.11 and Lemma 2.14. In the general case, we have to prove that every function g on W can uniquely be written as a polynomial in w of degree $d - 1$ over $\mathcal{O}_X$.

To show existence, note that we can find a neighborhood V of Y which is open in $\mathbb{C}^n$ such that P extends to V, and cuts out a definable analytic space W_V. Shrinking further and using Lemma 2.14 we may assume that g extends to W_V, and so it can be written as a polynomial in w of degree $d - 1$ over $\mathcal{O}_V$. Restricting to Y proves existence. Uniqueness is true in the analytic category (see e.g. [17, pg.56]) so follows from Theorem 2.22.

Now, by Corollary 2.15 it follows that pushforwards under finite maps are exact, and thus pushforwards of coherent sheaves from W to Y are coherent, and commute with analytification. Let $i : Z \to W$ be the natural inclusion map. It is clear that $i_*\mathcal{O}_Z$ is cut out by the ideal sheaf of Z restricted to W, and is therefore coherent, and analytifies to $i^{\mathrm{an}}_*\mathcal{O}_Z$. Thus we see that $\pi_*\mathcal{O}_Z$ is a coherent sheaf on Y, and analytifies to $\pi^{\mathrm{an}}_*\mathcal{O}_{Z^{\mathrm{an}}}$ and so the claim follows as above by Corollary 2.15. □

Lemma 2.36. *Let $\pi : X \to Y$ be a proper map of definable analytic spaces that is surjective on points and such that $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ is injective. Let $f \in \Gamma(Y^{\mathrm{an}}, \mathcal{O}_Y^{\mathrm{an}})$ be such that $\pi^*f \in \Gamma(X, \mathcal{O}_X)$. Then $f \in \Gamma(Y, \mathcal{O}_Y)$.*

Proof. By Lemma 2.4 we get a map $X \times \mathbb{C}$ corresponding to the section π^*f. Thus we get a map $\varphi : X \to Y \times \mathbb{C}$ whose projection to Y is the map π, such that the pullback of the $\mathbb{C}$-coordinate w is π^*f on X. Let Z be the set theoretic image of φ. It is clear that $Z \to Y \times \mathbb{C}$ is finite, and that Z is a definable holomorphic subset of $Y \times \mathbb{C}$, so we may give it its reduced induced structure by Lemma 2.28. Now let I_Z be the coherent ideal sheaf of Z in $Y \times \mathbb{C}$. The pullback π^*I_Z is a nilpotent coherent sheaf on X and thus some power of it is 0 by Theorem 2.33. Say $\pi^*I_Z^k = 0$. Set $Z_k \subset Y \times \mathbb{C}$ to be the definable analytic space cut out by I_Z^k. Then the map π factors through Z_k, and thus the map $\psi : Z_k \to Y$ is surjective on points, with the natural map $\mathcal{O}_Y \to \psi_*\mathcal{O}_{Z_k}$ being injective. By Lemma 2.35 we see that $\psi_*\mathcal{O}_{Z,k}$ is a coherent sheaf. Now $w \in \Gamma(Y, \psi_*\mathcal{O}_{Z_k})$ is in the image of $f \in \Gamma(\mathcal{O}_Y^{\mathrm{an}})$, and so the claim follows by Lemma 2.22. □

Corollary 2.37. *Let X, Y, Z be definable analytic spaces and suppose we have (solid) diagrams*

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle g}\downarrow & \swarrow{\scriptstyle i} & \\ Z & & \end{array} \qquad\qquad \begin{array}{ccc} X^{\mathrm{an}} & \xrightarrow{h^{\mathrm{an}}} & Y^{\mathrm{an}} \\ {\scriptstyle g^{\mathrm{an}}}\downarrow & \swarrow{\scriptstyle \iota} & \\ Z^{\mathrm{an}} & & \end{array}$$

such that h is surjective on points and $\mathcal{O}_Y \to h_*\mathcal{O}_X$ is injective. Then i exists such that $i^{\mathrm{an}} = \iota$.

Proof. It follows from definable choice that ι is a map of definable spaces. Let $U \subset Z$ be definable open and $f \in \mathcal{O}_Z(U)$. Then by Lemma 2.36 the section $\iota^*f \in \Gamma(\iota^{-1}(U), \mathcal{O}_{Y^{\mathrm{an}}})$ is actually in $\Gamma(\iota^{-1}(U), \mathcal{O}_Y)$. We thus get a map $i : Y \to Z$ and it is follows from Theorem 2.22 that $g = i \circ f$. □

2.10. **Quotients by finite groups.** To definabilize $\Gamma\backslash\Omega$ when Γ is not torsion-free, we shall need to be able to talk quotients of definable analytic spaces by finite groups. To that end, we have the following proposition:

Proposition 2.38. *Let X be a definable analytic space, and G a finite group acting on X. Then there exists a definable analytic space Y and a map $Q : X \to Y$ such that any map of definable analytic spaces $X \to Z$ which is G-invariant factors uniquely through Q. Moreover, Q analytifies to the analytic quotient, so that $\mathcal{O}_{Y,Q(x)} = \mathcal{O}_{X,x}^{I_x}$ where I_x is the stabilizer of x.*

Proof. It is well known that definable spaces admit quotients by proper equivalence relations, so let Y be the definable space which is the quotient of X by the proper equivalence relation given by G-equivalence. We need to equip Y with a structure sheaf giving it the structure of a definable analytic space. For each subgroup $H < G$, let X_H be the subset of X with stabilizer H, and let Y_H be the image of X_H in Y. Note that Y_H is determined by H up to conjugacy, and that G/H defines an étale equivalence relation on X_H. By Lemma 2.18 we may find a cover U_i of X_H in X by basic definable analytic varieties, such that the equivalence relation on $U_i \cap X_H$ is trivial. Replacing each U_i by its intersection with its H-translates we can assume that U_i is H-invariant. By picking a definable distance function, we further replace U_i by the set of points that are closer to $U_i \cap X_H$ then any of its translates.We may thus assume that if $z, gz \in U_i$ then $g \in H$.

Let V_i be the image of U_i in Y. Let $s_1, \ldots, s_m$ be coordinates for U_i. Consider the free polynomial ring $\mathbb{C}[h(s_i)]$ for $h \in H$ and let P_j be polynomials in the $(h(s_i))$ which generate the H-invariant subring. Then the P_j descend to analytic functions on V_i which embed it as a locally closed subset of $\mathbb{C}^n$, and thus give V_i the structure of a definable analytic space. We claim that $\pi : U_i \to V_i$ is the categorical quotient of U_i by H, and thus that V_i is the categorical quotient of $\pi^{-1}(U_i)$ by G.

To see this, suppose $f : U_i \to Z$ is a map of definable analytic spaces which is H invariant. Then f^{an} factors through π^{an}, so the statement follows from Corollary 2.37. It follows that the U_i glue to give a definable analytic space structure on Y, and that Y is the categorical quotient of X in the category of definable analytic spaces. □

3. Definable GAGA

In this section we prove an algebraization theorem for definable coherent sheaves on algebraic spaces. Precisely, we show:

Theorem 3.1. *Let X be an algebraic space and* $\mathrm{Def} : \mathbf{Coh}(X) \to \mathbf{Coh}(X^{\mathrm{def}})$ *the definibilization functor. Then*

(1) Def *is fully faithful and exact.*
(2) The essential image of Def *is closed under subobjects and quotients.*

Example 3.2. Def is *not* essentially surjective. Let q be the standard coordinate on $\mathbf{G}_m$. Note that the $\mathbb{C}$-local system V on $\mathbf{G}_m^{\mathrm{an}}$ with mondromy $\lambda = e^{2\pi i\alpha}$ can be trivialized on a definable open cover—take for instance a finite union of overlapping sectors. It follows that $\mathcal{F} = V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbf{G}_m^{\mathrm{def}}}$ is a definable coherent sheaf. On the one hand, the only algebraic line bundle on $\mathbf{G}_m$ is the trivial bundle $\mathcal{O}_{\mathbf{G}_m}$.

On the other hand, we claim that $\mathcal{F}$ can be nontrivial as a definable coherent sheaf. In the structure $\mathbb{R}_{\mathrm{alg}}$ this is obvious, as sections of $\mathcal{F}$ have at most finite monodromy, so if α is irrational $\mathcal{F}$ cannot be trivialized. Even in the structure $\mathbb{R}_{\mathrm{an}}$, however, $\mathcal{F}$ will be nontrivial if α is not real. A trivializing section is of the form $v \otimes e^{-\alpha \log q + g(q)}$ for holomorphic g, but as definable functions in $\mathbb{R}_{\mathrm{an}}$ grow sub-exponentially g must be constant. But $e^{-\alpha \log q}$ is clearly not definable on any sector if α is not real.

Before the proof we make some preliminary observations. First, we have the analog of Theorem 2.22.

Lemma 3.3. Def *is faithful and exact.*

Proof. By Lemma 2.22 the map $\mathrm{An} : \mathbf{Coh}(X^{\mathrm{def}}) \to \mathbf{Coh}(X^{\mathrm{an}})$ is faithful and exact. By classical considerations, the usual analytification functor $\mathrm{An} \circ \mathrm{Def}$ is faithful and exact. It follows that Def is also faithful and exact. □

Now, observe that given Lemma 3.3, part (2) of Theorem 3.1 implies (1) since we just need to verify fullness, and a homomorphism $F_1 \to F_2$ can be recovered from its graph as a subsheaf of $F_1 \oplus F_2$. Moreover, the first part of (2) clearly implies the second part by considering the kernel and using the exactness part of Lemma 3.3.

3.1. Vector bundles.

We first show a special case of Theorem 3.1. Suppose X is a reduced algebraic space, and let F be a coherent locally free sheaf on X.

Lemma 3.4. *If $0 \to \mathcal{E} \to F^{\mathrm{def}} \to \mathcal{G} \to 0$ is an exact sequence in $\mathbf{Coh}(X^{\mathrm{def}})$ and $\mathcal{E}$ and $\mathcal{G}$ are locally free, then it is the image by* Def *of an exact sequence $0 \to E \to F \to G \to 0$ in $\mathbf{Coh}(X)$ where E and G are locally free.*

Proof. It is sufficient to construct the quotient G and then define E as the kernel of $F \to G \to 0$. By working separatly on every connected component of X, one can assume that $\mathcal{G}$ have constant rank r. Let $\mathrm{Gr}(r,F)$ be the Grassmannian of quotient modules of F that are locally free of rank r. Then $\mathcal{G}$ corresponds to a definable section of $\mathrm{Gr}(r,F)^{\mathrm{def}}$, which is necessarily algebraic by[4] [34, Corollary 4.5].

□

3.2. **Proof of Theorem 3.1.** As above, it is enough to prove the first part of (2).

Lemma 3.5. *Assume Theorem 3.1 for reduced algebraic spaces. Then it is true for nonreduced spaces.*

Proof. Let X be a scheme with a nilpotent ideal I cutting out a subscheme X_0. We can assume $I^2=0$ and inductively that Theorem 3.1 holds for X_0.

Let F be a coherent sheaf on X and $\mathcal{E}\subset F^{\mathrm{def}}$ a definable coherent subsheaf. We have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (IF)^{\mathrm{def}} & \longrightarrow & F^{\mathrm{def}} & \longrightarrow & (F/IF)^{\mathrm{def}} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I^{\mathrm{def}}\mathcal{E} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}/I^{\mathrm{def}}\mathcal{E} & \longrightarrow & 0 \end{array}$$

Since Theorem 3.1 holds for X_0, the left vertical arrow is algebraic. Thus $I^{\mathrm{def}}\mathcal{E}=M^{\mathrm{def}}$ for a coherent $M\subset F$. We may thus replace F by F/M, and reduce to the case $I^{\mathrm{def}}\mathcal{E}=0$. Likewise, $\mathcal{E}$ maps to $(F/IF)^{\mathrm{def}}$ and must have algebraic image N^{def} for a coherent $N\subset F/IF$. Replacing F by the inverse image of N, we may assume that $\mathcal{E}$ maps isomorphically to $(F/IF)^{\mathrm{def}}$. Thus we are reduced to showing that if $F\to(F/IF)$ has a definable section then it is algebraic. Note that this section would have to land in P^{def}, where $P\subset F$ is the subsheaf annihilated by I. Since both F/IF and P are both coherent sheaves on X_0, this follows from Theorem 3.1 for X_0. □

We now assume X is reduced. Let F be a coherent sheaf on X and $\mathcal{E}\subset F^{\mathrm{def}}$ a definable coherent subsheaf.

Lemma 3.6. *For some dense open $U\subset X$, $\mathcal{E}_{|U}$ is algebraic.*

Proof. On some dense open set U, F is locally free. The (reduced) locus where $\mathcal{E}$ and $F^{\mathrm{def}}/\mathcal{E}$ have non-maximal rank is definable, analytic, and closed, hence algebraic by [34, Corollary 4.5]. After possibly shrinking U to a smaller dense open set, the claim then follows from Lemma 3.4.

□

Let E_U be the algebraic sheaf on U for which $(E_U)^{\mathrm{def}}\cong\mathcal{E}_{|U}$. Let $\widetilde{E}$ be the "closure" of E_U in F, i.e. the pullback

$$\begin{array}{ccc} F & \longrightarrow & j_*j^*F \\ \uparrow & & \uparrow \\ \widetilde{E} & \longrightarrow & j_*E_U \end{array}$$

where $j:U\hookrightarrow X$ denotes the inclusion. The sheaf $\widetilde{E}$ is evidently quasi-coherent and so it is coherent since it is a subsheaf of F. Thus, $\widetilde{E}^{\mathrm{def}}$ and $\mathcal{E}$ are both definable coherent subsheaves of F^{def}, and therefore so is their intersection $\mathcal{G}$.

Let I_Z be the ideal sheaf of $Z=X\smallsetminus U$ with the reduced scheme structure, and $\mathcal{I}=I_Z^{\mathrm{def}}$.

[4]See also the version in [30, Theorem 2.2]. Note that the statement can easily be generalized to reduced algebraic spaces.

Lemma 3.7. *Suppose we have definable coherent sheaves $\mathcal{G} \subset \mathcal{G}'$ for which $\mathcal{G}_{|U} = \mathcal{G}'_{|U}$. Then for some n, $\mathcal{I}^n \mathcal{G}' \subset \mathcal{G}$.*

Proof. Take the quotient

$$0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{Q} \to 0.$$

By Lemma 2.33, $\mathcal{I}^n$ kills $\mathcal{Q}$ for some n, and thus $\mathcal{I}^n \mathcal{G} \subset \mathcal{G}'$. □

Applying the lemma to $\mathcal{G} \subset \widetilde{E}^{\mathrm{def}}$, we have $(I_Z^n \widetilde{E})^{\mathrm{def}} \subset \mathcal{E}$. The quotient $\mathcal{E}'$ is then a subsheaf of $(F')^{\mathrm{def}}$, where $F' = F/I_Z^n \widetilde{E}$ is supported on a subspace whose reduction is Z. By induction, $\mathcal{E}'$ is algebraic, and $\mathcal{E}$ is the preimage in F, hence algebraic, so the proof is complete.

3.3. **Definable Chow.** We therefore obtain a version of the definable Chow theorem of Peterzil–Starchenko [34, Corollary 4.5] for arbitrary algebraic spaces.

Corollary 3.8. *Let Y be an algebraic space and $\mathcal{X} \subset Y^{\mathrm{def}}$ a closed definable analytic subspace. Then $\mathcal{X}$ is (uniquely) the definabilization of an algebraic subspace.*

Proof. We need only algebraize the quotient $\mathcal{O}_Y^{\mathrm{def}} \to \mathcal{O}_{\mathcal{X}}$, which follows from Theorem 3.1. □

Corollary 3.9. *Let X, Y be algebraic spaces. Then any map $X^{\mathrm{def}} \to Y^{\mathrm{def}}$ of definable analytic spaces is (uniquely) the definabilization of an algebraic map.*

Proof. Apply the previous corollary to the graph. □

4. Definable images

The purpose of this section is to prove an algebraization theorem for definable images of algebraic spaces. For convenience we make the following definition.

Definition 4.1. A map $f : X \to Y$ of algebraic spaces is dominant if $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is injective.

Note that a proper dominant map is surjective on complex points. Our goal is to prove the following result.

Theorem 4.2. *Let X be an algebraic space, $\mathcal{S}$ a definable analytic space, and $\varphi : X^{\mathrm{def}} \to \mathcal{S}$ a proper definable analytic map. Then φ (uniquely) factors as $\varphi = i \circ f^{\mathrm{def}}$ for a proper dominant map $f : X \to Y$ of algebraic spaces and a closed immersion $i : Y^{\mathrm{def}} \to \mathcal{S}$. Moreover, $i^{\mathrm{an}}(Y^{\mathrm{an}})$ coincides with the image $\varphi^{\mathrm{an}}(X^{\mathrm{an}})$.*

Remark 4.3. We expect a constructible analogue of this theorem to hold even if the map is not proper.

The proof of Theorem 4.2 will crucially use the following proposition:

Proposition 4.4. *Let $f : W \to Z$ be a proper dominant map of algebraic spaces. Suppose we have an algebraic square-zero thickening $W \to W'$, a definable square-zero thickening $Z^{\mathrm{def}} \to \mathcal{Z}'$, and a map $\varphi' : W'^{\mathrm{def}} \to \mathcal{Z}'$ which fits into a commutative diagram*

$$\begin{array}{ccc} W^{\mathrm{def}} & \longrightarrow & W'^{\mathrm{def}} \\ {\scriptstyle f^{\mathrm{def}}}\downarrow & & \downarrow{\scriptstyle \varphi'} \\ Z^{\mathrm{def}} & \longrightarrow & \mathcal{Z}' \end{array}$$

Then there are uniquely the following: a (proper) dominant map $f' : W' \to Z''$ of algebraic spaces, $Z \to Z''$ an algebraic square-zero thickening, and $Z''^{\mathrm{def}} \to \mathcal{Z}'$ a definable square-zero thickening such that we have commutative diagrams

$$
\begin{array}{ccc}
W & \longrightarrow & W' \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
Z & \longrightarrow & Z''
\end{array}
\qquad\qquad
\begin{array}{ccc}
W'^{\mathrm{def}} & & \\
 & \searrow^{\varphi'} & \\
{\scriptstyle f'^{\mathrm{def}}}\downarrow & & \mathcal{Z}' \\
 & \nearrow & \\
Z''^{\mathrm{def}} & &
\end{array}
\quad .
$$

Proof that Proposition 4.4 implies Theorem 4.2. We proceed by induction on the dimension. Assume first that X is reduced. It is enough to prove the theorem for each irreducible component X_i of X, as $X \to Y$ is obtained as the pushout of the $X_i \to Y_i$ guaranteed by the theorem. We therefore assume X irreducible in addition. Since φ is proper, $\varphi(X)$ is definable and analytic, so we may replace $\mathcal{S}$ by the reduced definable analytic space $\varphi(X)$ and thus assume that $\varphi : X^{\mathrm{def}} \to \mathcal{S}$ is surjective on points.

We will first explain how to reduce to the case where φ is a proper modification. Let $\mathrm{Hilb}(X)$ be the Hilbert space[5] of proper algebraic subspaces of X. Let H be the union of the components which contain the general fibers of the map $X^{\mathrm{def}} \to \mathcal{S}$, and $Z_H \subset X \times H$ the universal subscheme. Since φ is flat[6] over a definable Zariski open subset of $\mathcal{S}$, the fibers over this subset form a subset $U \subset H$ which is constructible in the definable analytic category, and therefore also in the algebraic category [30, Corollary 2.3]. Let H' be the closure of U in H, which is a closed reduced algebraic subspace of H. We claim that the fibers of $Z_{H'}$ over H' map (set-theoretically) to points in $\mathcal{S}$. Indeed, if $\xi_i \in U$ is a sequence converging to $\xi \in H$, for any $z \in Z_\xi$ we can choose $z_i \in Z_{\xi_i}$ converging to z, but the sequence $\varphi(Z_{\xi_i})$ has a unique limit. Thus, after taking normalizations $\tilde{Z}_{H'}$ and $\tilde{H}'$ of $Z_{H'}$ and H', respectively, we obtain a factorization

$$
\begin{array}{ccc}
 & \tilde{Z}_{H'} & \\
\swarrow & & \searrow \\
X & & \tilde{H}'
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & (\tilde{Z}_{H'})^{\mathrm{def}} & \\
\swarrow & & \searrow \\
X^{\mathrm{def}} & & (\tilde{H}')^{\mathrm{def}} \\
\searrow & & \swarrow \\
 & \mathcal{S} &
\end{array}
$$

Note that we have used Corollary 2.37 to ensure $(\tilde{H}')^{\mathrm{an}} \to \mathcal{S}^{\mathrm{an}}$ is definable. By construction $(\tilde{H}')^{\mathrm{def}} \to \mathcal{S}$ is a proper modification (indeed, it is one-to-one on $\tilde{U}'$), and it will be enough to algebraize $(\tilde{H}')^{\mathrm{def}} \to \mathcal{S}$, as the algebraicity of $X^{\mathrm{def}} \to \mathcal{S}$ then follows from [34, Corollary 4.5].

We may therefore assume $X^{\mathrm{def}} \to \mathcal{S}$ is a modification. Now, by induction, the exceptional locus of $X^{\mathrm{def}} \to \mathcal{S}$ can be algebraized, so let $Z^{\mathrm{def}} \subset \mathcal{S}$ be the reduced exceptional locus, and $W = \varphi^{-1}(Z)$ equipped with its reduced induced structure. Let W_k be the k'th order thickening of W, and $\mathcal{Z}_k$ the k'th order thickening of Z^{def} in $\mathcal{S}$.

We claim that $W_k^{\mathrm{def}} \to \varphi(W_k^{\mathrm{def}})$ can be algebraized. Indeed, by induction $W_{k-1}^{\mathrm{def}} \to \varphi(W_{k-1}^{\mathrm{def}})$ can be algebraized, and so we get a commutative diagram

[5]Alternatively, one can first pass via a proper modification $X' \to X$ to a scheme X' and then deal with the usual Hilbert scheme.

[6]More directly, after replacing X with a resolution, φ is smooth over a definable Zariski open.

$$\begin{array}{ccc} W_{k-1}^{\mathrm{def}} & \longrightarrow & W_k^{\mathrm{def}} \\ \downarrow & & \downarrow \\ \varphi(W_{k-1})^{\mathrm{def}} & \longrightarrow & \mathcal{Z}_k \end{array}$$

and applying Proposition 4.4 algebraizes $W_k^{\mathrm{def}} \to \varphi(W_k^{\mathrm{def}})$.

We thus form a formal algebraic space $\overline{Z}$ to which the completion $\overline{W}$ of W inside X maps. Since $X^{\mathrm{def}} \to \mathcal{S}$ is a proper modification, it follows exactly as in the proof [2, Lemma 7.7] that the map $\overline{W} \to \overline{Z}$ is a formal modification, and thus $X^{\mathrm{an}} \to \mathcal{S}^{\mathrm{an}}$ can be algebraized as $X \to S$ by [2, Theorem 3.1]. Now it remains to show that $S^{\mathrm{def}} = \mathcal{S}$, or said differently that the algebraic functions on S are definable with respect to the definable structure on $\mathcal{S}$. This follows immediately from Corollary 2.36.

Now if X is non-reduced, $X^{\mathrm{an}} \to \varphi^{\mathrm{an}}(X^{\mathrm{an}})$ can be algebraized by $X \to Y$ exactly as above by repeatedly applying Proposition 4.4. It remains to show that the map $Y^{\mathrm{an}} \to \mathcal{S}^{\mathrm{an}}$ is the analytification of a map $Y^{\mathrm{def}} \to \mathcal{S}$. To see this, first note that the map on spaces is definable, since it is the unique map through which $X^{\mathrm{def}} \to \mathcal{S}$ factors. The claim now follows from Lemma 2.37. □

We now prove Proposition 4.4. We have an exact sequence of $\mathcal{O}_{W'}$-coherent sheaves on W':

$$0 \to I \to \mathcal{O}_{W'} \to \mathcal{O}_W \to 0 \tag{1}$$

where both I and $\mathcal{O}_W$ are coherent $\mathcal{O}_W$-sheaves. We can analytify on W' and get a sequence of sheaves of $\mathcal{O}_{W'^{\mathrm{an}}}$-coherent sheaves[7] :

$$0 \to I^{\mathrm{an}} \to \mathcal{O}_{W'^{\mathrm{an}}} \to \mathcal{O}_{W^{\mathrm{an}}} \to 0. \tag{2}$$

Viewing (1) in the category of sheaves of abelian groups we have a natural coboundary map $f_*\mathcal{O}_W \to R^1 f_* I$, while viewing (2) in the category of $\mathcal{O}_{W'^{\mathrm{an}}}$-modules we have a coboundary map $f_*^{\mathrm{an}}\mathcal{O}_{W^{\mathrm{an}}} \to R^1 f_*^{\mathrm{an}} I^{\mathrm{an}}$.

Lemma 4.5. *The coboundary map $f_*\mathcal{O}_W \to R^1 f_* I$ analytifies to the coboundary map $f_*^{\mathrm{an}}\mathcal{O}_{W^{\mathrm{an}}} \to R^1 f_*^{\mathrm{an}} I^{\mathrm{an}}$.*

Proof. The boundary map factors through Cech cohomology, and the statement follows. □

Let F be the kernel of $f_*\mathcal{O}_W \to R^1 f_* I$, so that by the preceding lemma F is an $\mathcal{O}_Z$-module. Note that analytically, it is clear that $\mathcal{O}_{Z'^{\mathrm{an}}}$ surjects onto $\mathcal{O}_{Z^{\mathrm{an}}}$, and thus the image of $\varphi'^{\mathrm{an}}_*\mathcal{O}_{W'^{\mathrm{an}}}$ in $f_*^{\mathrm{an}}\mathcal{O}_W^{\mathrm{an}}$ contains the image of $\mathcal{O}_{Z^{\mathrm{an}}}$. It follows that F^{an} contains the image of $\mathcal{O}_{Z^{\mathrm{an}}}$ in $f_*\mathcal{O}_W^{\mathrm{an}}$, and since these are coherent sheaves it follows by Theorem 2.22 that F contains the image of $\mathcal{O}_Z$ in $f_*\mathcal{O}_W$. We define the sheaf of rings R on Z as the pushout $R = \mathcal{O}_Z \oplus_{f_*\mathcal{O}_W} f_*\mathcal{O}_{W'}$. It follows that R surjects onto $\mathcal{O}_Z$, with nilpotent kernel $J = f_* I$.

Lemma 4.6. *Suppose Z is an algebraic space, and J is a quasi-coherent sheaf on Z. Let R be a sheaf of rings on the étale site $\underline{Z}$ of Z such that*

$$0 \to J \to R \to \mathcal{O}_Z \to 0$$

is a first order thickening[8]*. Then $(\underline{Z}, R)$ is an algebraic space.*

[7] Note that $\mathcal{O}_{W'^{\mathrm{an}}}$, when viewed on W, is not the analytification of $\mathcal{O}_{W'}$ since that is not defined (it's not even an $\mathcal{O}_W$-module).

[8] Recall this means that $R \to \mathcal{O}_Z$ preserves the rings structure and that J with its induced ideal structure is of square zero.

Proof. The statement is étale local, so we may assume Z is a scheme. Let $\psi : R \to \mathcal{O}_Z$ be the natural map. Since R is a nilpotent thickening of $\mathcal{O}_Z$ it follows that it has the same spectrum. Thus it suffices to show that if $U \subset Z$ is affine, and $f \in R(U)$, that $\Gamma(U, R)_f$ maps isomorphically to $\Gamma(U_f, R)$.

We first check injectivity. Suppose $s \in \Gamma(U, R)$, and restricts to 0 in U_f. Then replacing s by $f^m s$ for some positive integer m we may assume $\psi(s) = 0$. Thus $s \in \Gamma(U, J)$. But J is a quasi-coherent sheaf, thus $f^r s = 0$ for some positive integer r, as desired.

Conversely, suppose $s \in \Gamma(U_f, R)$. Replacing s by $f^m s$ we may assume $\psi(s)$ extends, and thus subtracting off that $s \in \Gamma(U_f, J)$, and again the statement follows by quasi-coherence of J. □

We thus have an algebraic thickening $Z_0 = (\underline{Z}, R)$ and a diagram

$$\begin{array}{ccc} W & \longrightarrow & W' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z_0 \end{array}$$

where $W' \to Z_0$ is dominant, since f is dominant. Consider the diagram

$$\begin{array}{ccccc} W^{\mathrm{def}} & \longrightarrow & W'^{\mathrm{def}} & & \\ \downarrow & & \downarrow & \searrow & \\ & & & & \mathcal{Z}' \\ \downarrow & & \downarrow & \nearrow & \\ Z^{\mathrm{def}} & \longrightarrow & Z_0^{\mathrm{def}} & & \end{array}$$

Note that $Z_0^{\mathrm{def}} \to \mathcal{Z}'$ may well not be immersive. We claim that the image is algebraic. The definable analytic space structure on the image is defined by the image $\mathcal{T}$ of the map $\mathcal{O}_{\mathcal{Z}'} \to R^{\mathrm{def}}$, and we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J^{\mathrm{def}} & \longrightarrow & R^{\mathrm{def}} & \longrightarrow & \mathcal{O}_{Z^{\mathrm{def}}} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{O}_{Z^{\mathrm{def}}} & \longrightarrow & 0 \end{array}$$

Now $\mathcal{K}$ is a coherent $\mathcal{O}_{Z^{\mathrm{def}}}$-submodule of J^{def} and therefore the definabilization of an algebraic $K \subset J$ by Theorem 3.1. Letting $R' = R/K$, we have another algebraic space $(\underline{Z}, R')$ by Lemma 4.6. Note that $\mathcal{T}/K^{\mathrm{def}} \cong \mathcal{O}_{Z^{\mathrm{def}}}$ is a coherent R'^{def}-submodule of R'^{def} and therefore gives a copy of $\mathcal{O}_Z$ inside R', by Theorem 3.1. It is easy to check that the pushout $T = R \oplus_{R'} \mathcal{O}_Z$ definabilizes to $\mathcal{T}$, and by Lemma 4.6, $Z'' = (\underline{Z}, T)$ is algebraic.

Since $(Z^{\mathrm{def}}, \mathcal{T})$ is the image of φ' by construction, this concludes the proof.

5. Algebraicity of period maps

In this section we prove the first part Theorem 1.1. **For this section we work only over the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$.**

5.1. **Period images.** Let Ω be a pure polarized period domain with generic Mumford–Tate group $\mathbf{G}$ and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ an arithmetic lattice. By [3, Theorem 1.1], if Γ is neat then $\Gamma\backslash\Omega$ has a canonical structure of a definable analytic variety (in fact, even over $\mathbb{R}_{\mathrm{alg}}$). Since every arithmetic lattice has a normal neat subgroup Γ', using Proposition 2.38 we can equip $\Gamma\backslash\Omega$ with a definable analytic space structure as the categorial quotient of $\Gamma'\backslash\Omega$ by $G = \Gamma/\Gamma'$.

Corollary 5.1. *Let X be a reduced algebraic space, and $\varphi : X^{\mathrm{an}} \to (\Gamma\backslash\Omega)^{\mathrm{an}}$ a locally liftable map satisfying Griffiths transversality. Then φ (uniquely) factors as $\varphi = \iota^{\mathrm{an}} \circ f^{\mathrm{an}}$ for a dominant map $f : X \to Y$ of algebraic spaces and a closed immersion $\iota : Y^{\mathrm{def}} \to \Gamma\backslash\Omega$ of definable analytic varieties.*

Proof. Taking a resolution, its enough to assume X is smooth, and by a theorem of Griffiths [20, Theorem 9.5] we may then assume that φ is proper. By [3, Theorem 1.3], $\varphi : X^{\mathrm{an}} \to (\Gamma\backslash\Omega)^{\mathrm{an}}$ is the analytification of a map $X^{\mathrm{def}} \to \Gamma\backslash\Omega$ of definable analytic varieties. Now apply Theorem 4.2. □

In fact, we obtain a version of 5.1 over non-reduced bases, but we must first make the following definition.

Definition 5.2. Let X be an algebraic space (possibly non-reduced). A definable period map of X is a locally liftable map $\varphi : X^{\mathrm{def}} \to \Gamma\backslash\Omega$ of definable analytic spaces such that for any reduced irreducible component X_0 of X, the associated (locally liftable definable) map $\varphi_0 : X_0^{\mathrm{def}} \to \Gamma\backslash\Omega$ satisfies Griffiths transversality—that is, the (locally defined) map $T_{X_0^{\mathrm{def}}} \to \varphi_0^* T_\Omega$ on the tangent sheaf $T_{X_0} = (\Omega^1_{X_0})^\vee$ factors through the Griffiths transverse subbundle.

Note that we do not require φ to be Griffiths transverse in the nilpotent tangent directions. Moreover, note that the definition is functorial in the sense that for any definable period map $\varphi : X^{\mathrm{def}} \to \Gamma\backslash\Omega$ and any map $f : Y \to X$, we have that $f \circ \varphi$ is a period map. Finally, for X integral, the Griffiths transversality condition is equivalent to the usual condition on the regular locus $X^{\mathrm{reg}} \subset X$.

The local liftability condition is equivalent to φ factoring through the stack quotient $[\Gamma\backslash\Omega]$ which is naturally a definable analytic Deligne–Mumford stack using the proof of Proposition 2.38. There are no new subtleties in the definition of a definable analytic Deligne–Mumford stack, but we do not pursue these ideas here. Note that by definable cell decomposition (as in Lemma 2.14), φ is definably locally liftable if and only if it is analytically locally liftable.

The definability requirement should be viewed as an admissibility condition. The following example shows Corollary 5.1 is in general false without assuming definability, while Proposition 5.9 below shows that period maps associated to variations coming from algebraic families are definable.

Example 5.3. Let $S^{\mathrm{an}} = \Gamma\backslash\Omega$ be a modular curve with level structure so that it is a smooth scheme, and let $Y = S \times_{\mathrm{Spec}\,\mathbb{C}} \mathrm{Spec}\,\mathbb{C}[\epsilon]/(\epsilon^2)$ be the trivial thickening of it. Given a global holomorphic function f on S^{an} and a global derivation D on S we can define a map $\varphi : Y^{\mathrm{an}} \to S^{\mathrm{an}}$ extending the identity map via $\varphi^\sharp(s) = s + \epsilon f Ds$. Since S is affine we can pick f to be non-algebraic, and then the map φ will be non-algebraizable.

With these preliminaries, we now state a more general version of Corollary 5.1, to be proven in the next subsection.

Theorem 5.4. *Let X be an algebraic space and $\varphi : X^{\mathrm{def}} \to \Gamma\backslash\Omega$ a definable period map. Then φ (uniquely) factors as $\varphi = \iota \circ f^{\mathrm{def}}$ for a dominant map $f : X \to Y$ of algebraic spaces and a closed immersion $\iota : Y^{\mathrm{def}} \to \Gamma\backslash\Omega$ of definable analytic spaces.*

Definition 5.5. We refer to an algebraic space Y with a closed immersion $\iota : Y^{\mathrm{def}} \to \Gamma\backslash\Omega$ of definable analytic spaces arising from the theorem as a *definable period image.*

Note that for a *proper* definable period map $X^{\mathrm{def}} \to \Gamma\backslash\Omega$ Theorem 5.4 holds over an arbitrary o-minimal structure.

5.2. **Algebraicity of the Hodge filtration.** We make the following definition along the same lines as in the previous subsection:

Definition 5.6. Let Y be an algebraic space (possibly non-reduced). A definable variation on Y is a triple $(V_{\mathbb{Z}}, \mathcal{F}^\bullet, Q)$ where $V_{\mathbb{Z}}$ is a local system $V_{\mathbb{Z}}$ on Y^{def}, $\mathcal{F}^\bullet$ is a definable coherent locally split filtration of $V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_{Y^{\mathrm{def}}}$ satisfying Griffiths transversality (in the same sense as Definition 5.2), and Q is a quadratic form on $V_{\mathbb{Z}}$, such that $(V_{\mathbb{Z}}, \mathcal{F}^\bullet, Q)$ is a pure polarized integral Hodge structure fiberwise.

As above, every local system on Y^{an} is definable by o-minimal cell decomposition. If Γ is torsion-free the triple $(V_{\mathbb{Z}}, \mathcal{F}^\bullet, Q)$ exists universally on $\Gamma\backslash\Omega$ (although of course it is *not* in general a variation as it does not satisfy Griffiths transversality).

When Y carries a definable variation that's clear from context, we denote by $F^\bullet_{Y^{\mathrm{def}}}$ the filtered Hodge bundle. If Y is smooth (in particular reduced) with a log smooth compactification $\overline{Y}$ and the variation has unipotent monodromy at infinity, we know that $F^\bullet_{Y^{\mathrm{def}}}$ is in fact the definabilization of an algebraic filtered bundle $F^\bullet_Y$. In fact, it has a canonical extension (the Deligne canonical extension) as an algebraic filtered bundle $F^\bullet_{\overline{Y}}$, uniquely determined by condition that the connection have log poles with nilpotent residues.

We now show the following generalization of the second part of Theorem 1.1:

Theorem 5.7. *For Y an algebraic space with a definable variation, $F^\bullet_{Y^{\mathrm{def}}}$ is the definabilization of a (unique) algebraic filtered bundle $F^\bullet_Y$.*

Proof. First, observe that taking a suitable level cover, we get an étale cover $f : Y' \to Y$ such that the variation on Y'^{def} has unipotent monodromy at infinity. As $F^\bullet_{Y^{\mathrm{def}}}$ embeds in $f^{\mathrm{def}}_* F^\bullet_{Y'^{\mathrm{def}}}$, by Theorem 3.1 we may assume that the monodromy at infinity is unipotent.

Let Y_0 be the reduced space of Y. Y_0 can be resolved by successive blow-ups, and performing the same blow-ups on Y we obtain $X \to Y$ whose reduced space X_0 is smooth. By taking some compactification and again blowing up to resolve the reduced boundary, we obtain a compactification $\overline{X}$ of X whose reduced space is log smooth. The following lemma then implies $F^\bullet_{X^{\mathrm{def}}}$ is the definabilization of an algebraic filtered bundle $F^\bullet_X$ by ordinary GAGA.

Lemma 5.8. *Let $\overline{X}$ be a proper algebraic space, and D a closed subspace such that the reduced spaces $(\overline{X}_0, D_0)$ are a log smooth pair, and such that $X = \overline{X} \smallsetminus D$ has a definable variation with unipotent monodromy at infinity. There is a unique map $f : \widetilde{X} \to \overline{X}$ which is an isomorphism on reductions and over X, and minimal with respect to the following property: $F^\bullet_{X^{\mathrm{def}}}$ extends as a filtered vector bundle to $\widetilde{X}^{\mathrm{def}}$ and restricts to the Deligne canonical extension on the reduced space $(\overline{X}_0)^{\mathrm{def}}$.*

Proof. $\overline{X}_0^{\mathrm{def}}$ admits a definable cover by polydisks $P = \Delta^n$ such that X_0^{def} is locally $P^* = (\Delta^*)^m \times \Delta^{n-m}$. Let $\mathcal{R}$ be the restriction of the definable structure sheaf of $\overline{X}^{\mathrm{def}}$ to P. Since an analytic space is Stein iff its reduction is Stein, $(P, \mathcal{R})$ is a Stein space, and so we may and do choose lifts t_k of the coordinate functions z_k on the reduction (by possibly shrinking further, these lifts are also definable). Note that a surjective exponential map $\mathcal{R} \to \mathcal{R}^\times$ is still well defined with kernel $\mathbb{Z}^n$.

Now let q_k be a choice of logarithm of t_k for each k, definable on vertical strips, and $N_1, \ldots, N_m$ the nilpotent monodromy logarithms. We have a definable map $\varphi : (\Delta^*)^m \times \Delta^{n-m} \to \Gamma\backslash\Omega$, and so $\psi = \exp(-\sum q_k N_k)\varphi$ lifts definably to $\check{\Omega}$. By standard theory, the map on the reduction extends to P.

Let $i : X \to \overline{X}$ and $j : P^* \to P$ be the inclusions, and consider the sheaf $j_*j^*\mathcal{R}$ as a sheaf of rings on P. We have a pullback $\psi^*\mathcal{O}_{\check{\Omega}} \to j_*j^*\mathcal{R}$, and we take $\mathcal{T}$ to be the subsheaf of $j_*j^*\mathcal{R}$ generated by this image and $\mathcal{R}$. We first claim that $\mathcal{T}$ is definable coherent. Indeed, while $j_*j^*\mathcal{R}$ is not coherent, it is $j_*(\mathcal{O}_X^{\mathrm{def}}|_P)$ and therefore naturally filtered by the definable coherent sheaves $E^{\mathrm{def}}|_P$ for $E \subset i_*\mathcal{O}_X$ coherent. Thus, there is some such E so that the image of $\psi^*\mathcal{O}_{\check{\Omega}}$ and $\mathcal{R}$ are both contained in $E^{\mathrm{def}}|_P$, and therefore $\mathcal{T}$ is definable coherent. $\mathcal{T}$ is uniquely determined as the "minimal thickening" of X such that ψ extends, so we obtain a well-defined definable analytic space $\widetilde{\mathcal{X}} = (\overline{X}_0, \mathcal{T})$. By construction, $\mathcal{O}_{\mathcal{X}'}$ is a subsheaf of F^{def} for some coherent $E \subset i_*\mathcal{O}_X$, and is therefore algebraic by Theorem 3.1. Set $\widetilde{\mathcal{X}} = \widetilde{X}^{\mathrm{def}}$.

Now, we can pull back the Hodge filtration on $\check{\Omega}$ to get a definable filtered vector bundle $F^\bullet_{\widetilde{X}}$ on $\widetilde{X}$ extending $F^\bullet_X$. The gluing follows because the filtration on $\check{\Omega}$ is invariant under $\mathbf{G}(\mathbb{C})$. □

Note that $X \to Y$ may not be dominant, but its image Y'' is isomorphic to Y on a dense open set U. Let Z be a sufficiently thick nilpotent neighborhood of the complement of U and $A = Y'' \times_Y Z$. Then Y is naturally the pushout

$$\begin{array}{ccc} A & \longrightarrow & Y'' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

Now as $f : X \to Y''$ is proper dominant, $F^\bullet_{Y''^{\mathrm{def}}}$ embeds in $f_*(F^\bullet_X)^{\mathrm{def}}$, so it is the definabilization of some algebraic $F^\bullet_{Y''}$ by Theorem 3.1. Z has smaller dimension then Y, so by induction $F^\bullet_{Z^{\mathrm{def}}} = (F^\bullet_Z)^{\mathrm{def}}$ is algebraic, and $F^\bullet_{Y^{\mathrm{def}}}$ is the definabilization of the pushout of $F^\bullet_{Y''}$ and $F^\bullet_Z$. □

Proof of Theorem 5.4. Let X be an algebraic space and $\varphi : X^{\mathrm{def}} \to \Gamma\backslash\Omega$ a definable period map. The proof of the previous theorem implies we can produce a proper $X' \to X$ such that the definable period map of X' has unipotent monodromy at infinity and $X' \to X$ is dominant on some dense open set U of X. Moreover, we get a partial compactification $\overline{X}'$ which admits a definable proper period map $\overline{\varphi} : \overline{X}'^{\mathrm{def}} \to \Gamma\backslash\Omega$ restricting to that of X'. Applying Theorem 4.2 to $\overline{X}'$, we obtain $\overline{X}' \to Y'$ (proper) dominant and $Y'^{\mathrm{def}} \to \Gamma\backslash\Omega$ a closed immersion.

Let X'' be the image of X' in X, and let W be a sufficiently thick nilpotent neighborhood of the complement of U such that X is the pushout of W and X''. By induction we may apply Theorem 4.2 to W to obtain a dominant $W \to Z$ and a closed immersion $Z^{\mathrm{def}} \to \Gamma\backslash\Omega$. The sought for Y is then the pushout of Z and Y'. □

Every definable period map yields a definable variation by pulling back[9], and we conclude this subsection with a converse.

Proposition 5.9. *Let Y be an algebraic space. An analytic period map $\varphi : Y^{\mathrm{an}} \to (\Gamma\backslash\Omega)^{\mathrm{an}}$ associated to a definable variation is definable.*

Proof. Again we may produce a proper $X \to Y$ such that the pull back of the variation to X has unipotent monodromy at infinity, has smooth reduced space, and for which

[9]Pulling back from the stack that is.

$X \to Y$ is dominant on a dense open set U of Y. Let Z be a sufficiently thick nilpotent thickening of the complement of U, and let X' be the image of X in Y. By induction we may assume the claim for Z. It will be enough to show the claim for X, for then by Corollary 2.37 we have it for X', and then the period map of Y is the pushout of those of X' and Z.

Therefore, replacing Y with X, we may assume Y has smooth reduced space Y^{red}. From [3], there is a definable fundamental set Ξ for Γ such that the quotient map $\Xi \to \Gamma\backslash\Omega$ realizes $\Gamma\backslash\Omega$ as a definable analytic space as the quotient of Ξ by a closed definable equivalence relation. As above, the reduced period map $\varphi^{\mathrm{red}} : (Y^{\mathrm{red}})^{\mathrm{an}} \to (\Gamma\backslash\Omega)^{\mathrm{an}}$ is definable, so there is a definable open cover $\mathcal{Y}_i$ of Y^{def} such that we can choose lifts $\mathcal{Y}_i \to \Xi$ which are definable on reduced spaces. But $\check{\Omega}$ is a flag variety and maps $\mathcal{Y}_i \to \check{\Omega}$ are clearly definable if and only if $\mathcal{F}^\bullet|_{\mathcal{Y}_i}$ is definable, and this implies $\mathcal{Y}_i \to \Xi$ is definable. □

Thus a definable variation on Y is equivalent to a definable period map.

Corollary 5.10. *Let Y be an algebraic space. A period map associated to an algebraic subquotient of a variation $R^k f_*\mathbb{Z}$ for a smooth projective family $f : X \to Y$ is definable.*

Proof. In this case the filtered Hodge bundle is algebraic. □

5.3. **The Griffiths bundle.** For any algebraic space Y with a definable map $Y^{\mathrm{def}} \to \Gamma\backslash\Omega$ we denote by $L_{Y^{\mathrm{def}}}$ the pullback of the Griffiths $\mathbb{Q}$-bundle.

Lemma 5.11. *Let Y be a definable period image. Then $L_{Y^{\mathrm{def}}}$ is the definabilization of a (unique) algebraic $\mathbb{Q}$-bundle L_Y.*

Proof. By definition there is an algebraic space X with a definable period map factoring through Y^{def} such that $f : X \to Y$ is dominant. By a similar argument as in the proof of Theorem 5.7, by possibly thickening Y we may assume f is proper. By Theorem 5.7 the Griffiths bundle on X is the definabilization of an algebraic $\mathbb{Q}$-bundle L_X. As $L_{Y^{\mathrm{def}}}$ embeds in $f_*(L_X)^{\mathrm{def}}$, we are done by Theorem 3.1. □

6. Quasi-projectivity of period images

Let Y be a definable period image in $\Gamma\backslash\Omega$. From the last subsection, we know that the Griffiths $\mathbb{Q}$-bundle L_Y is algebraic. Before proving the second part of Theorem 1.1, we will need the following notion.

Definition 6.1. Assume Y is a definable period image. For Y reduced, we say a section s of L_Y^m *vanishes at the boundary* if the following condition holds: for some (hence any) period map $X^{\mathrm{def}} \to \Gamma\backslash\Omega$ factoring through Y such that X is smooth and the variation on X unipotent monodromy at infinity, s pulls backs to a section of $L_X^m(-D)$. Note that s vanishes at the boundary if and only if s^{red} does. We let $\Gamma_{van}(Y, L_Y^m) \subset \Gamma(Y, L_Y^m)$ denote the linear subspace of sections vanishing at the boundary.

Note that $\Gamma_{van}(Y, L_Y^m)$ is finite-dimensional for each m. We are now in a position to state the main result of this section:

Theorem 6.2. *Let Y be a definable period image. Then L_Y is ample on Y. Moreover, sections of some power L_Y^n which vanish at the boundary realize Y as a quasi-projective scheme.*

For the proof we will need the following:

Lemma 6.3. *Let X, Y be algebraic spaces and $f : X \to Y$ a proper dominant map. There is an algebraic subspace $S \subset Y$ supported on the locus where f is not an isomorphism, such that for any line bundle L on Y, a section $s \in \Gamma(X, f^*L)$ is in the image of $\Gamma(Y, L)$ iff its restriction $s|_T \in \Gamma(T, f^*L|_T)$ is in the image of $\Gamma(S, L|_S)$, where $T = f^{-1}S$.*

Proof. Let Q be to cokernel of the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ and S its scheme-theoretic support. Then we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_Y & \longrightarrow & f_*\mathcal{O}_X & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ & & \mathcal{O}_S & \longrightarrow & f_*\mathcal{O}_T & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

Tensoring by L and taking cohomology, the result follows. □

In the proof of the theorem, the positivity of the Griffiths bundle will be deduced from the special case of variations over smooth bases, where we have the following:

Lemma 6.4. *Let X be the complement of a normal crossing divisor D in a compact Kähler manifold $\overline{X}$. Consider a polarized real variation of Hodge structure over X with unipotent monodromies around D and let $L_{\overline{X}}$ be Deligne canonical extension of the associated Griffiths line bundle. Then $L_{\overline{X}}$ is a nef line bundle. Moreover, $L_{\overline{X}}$ is big if and only if the associated period map is generically immersive.*

Proof. Since every piece of the Hodge filtration comes equipped with a metric, the restriction to X of the Griffiths line bundle is naturally equipped with a metric h, which by Schmid norm estimates extends as a singular metric on $\overline{X}$ with zero Lelong numbers (see [9] for more details). It follows that $L_{\overline{X}}$ is a nef line bundle. In particular, $L_{\overline{X}}$ is big if and only if $c_1(L_{\overline{X}})^{\dim X} > 0$, cf. [8, Theorem 1.2]. By [27, Theorem 5.1] this number is computed by the Chern form of the hermitian line bundle $(L_{\overline{X}}, h)$:

$$c_1(L_{\overline{X}})^{\dim X} = \int_X (C_1(L_{\overline{X}}, h))^{\dim X}.$$

We conclude using that the real $(1,1)$-form $C_1(L_{\overline{X}}, h)$ is strictly positive at a point $x \in X$ if and only if the period map is immersive at x, cf. [20, Proposition 7.15]. □

Proof of Theorem 6.2. We first show it suffices to assume Γ is neat. Recall that by Zariski's main theorem, to show a line bundle is ample it is sufficient to show that it separates points. Take $\Gamma' \subset \Gamma$ be a normal, neat subgroup of finite index ℓ, with quotient G. Let Y be the period image of X, and Y' the period image of the level cover of X' in $\Gamma'\backslash\Omega$. Then we have a surjective, dominant, finite map $\pi : Y' \to Y$, and a group action G on Y' such that π is G-invariant. Let s be a local section of Y', we claim that $\mathrm{Nm}(s) := \prod_{g\in G} g(s)$ descends to Y. It is enough to work on stalks. Let $y \in Y$, $R = \mathcal{O}^{\mathrm{an}}_{Y,y}$ and $S = \mathcal{O}^{\mathrm{an}}_{Y',\pi^{-1}(y)}$. Let $s \in S$, then s lifts to a section s_U on some (G-invariant) open neighborhood U of $\pi^{-1}(y)$. Now $\mathrm{Nm}(s_U)$ is in $\mathcal{O}^G_{(\Gamma'\backslash\Omega)^an} = \mathcal{O}_{(\Gamma\backslash\Omega)^{\mathrm{an}}}$ and thus has an image $r \in R$, whose image in S is therefore $\mathrm{Nm}(s)$. We therefore have norm maps

$$\mathrm{Nm} : \Gamma(Y', L^k_{Y'}) \to \Gamma(Y, L^{k\ell}_Y)$$

for each k.

It follows from the assumption that $L_{Y'}$ is ample on Y' that L_Y separates points on Y. Thus, L_Y is ample.

Let $P \subset Y$ be a closed, zero-dimensional subscheme. We will show:

Claim. The restriction $\Gamma_{van}(Y, L^n_Y) \to \Gamma(P, L^n_P)$ is surjective for some n.

We now prove the claim by induction on $d = \dim Y$, the case of $d = 0$ being trivial.

Step 1. We first show we may assume Y is reduced. If Y is non-reduced, then we can write Y as a thickening of a subspace Y_0 by a square-zero sheaf of ideals I. Note that I is an $\mathcal{O}_{Y_0}$-module. We have an exact sequence

$$0 \to IL_Y^n \to L_Y^n \to L_{Y_0}^n \to 0. \tag{3}$$

By the induction statement, on Y_0 we can pick an embedding $Y_0 \to \mathbb{P}^m$ corresponding only to sections that vanish at the boundary. Thus, we can find a section $g \in \Gamma_{van}(Y_0, L_{Y_0}^n)$ such that Y_{0g} is affine and contains the base change P_0 of P. Now since Y_{0g} is affine, we may pick vanishing sections $s_1, \ldots, s_k$ of $L_{Y_0}^m$ whose image spans $\Gamma(P_0, L_{P_0}^n)$. By the Cech Cohomology argument in [22, II.4.5.14], it follows that for a sufficiently large integer r we have that $s_1 g^r, \ldots, s_k g^r$ all lift to sections of L_Y^{nr+m} on Y. Since g does not vanish on P, it follows that $\Gamma_{van}(Y, L_Y^n)$ surjects onto $\Gamma(P_0, L_{P_0}^n)$ for $n \gg 1$. Now from (3) it is enough to show that $\Gamma(Y_0, IL_Y^n)$ surjects onto $\Gamma(P_0, IL_P^n)$, which is an immediate consequence of L_{Y_0} being ample.

Step 2. By Step 1, we assume Y is reduced. Take a resolution $f : X \to Y$ which is an isomorphism outside of a dimension $d-1$ set and which has a log-smooth projective compactification $\overline{X}$. Let $S \subset Y$ and $T \subset X$ be the subspaces guaranteed by Lemma 6.3 for $L = L_{\overline{X}}$. Let Z be the pushout of T and $f^{-1}P$.

Step 3.

Lemma 6.5. *There is a (nonzero) effective divisor $\overline{E}$ in $\overline{X}$ containing Z such that for $m \gg 0$, any section of $\Gamma(\overline{E}, L_{\overline{X}}^m(-D)|_{\overline{E}})$ whose restriction to Z is a pullback, is the restriction of the pullback of a vanishing section of L_Y^m.*

Proof. Let A be an ample divisor on $\overline{X}$. $L_{\overline{X}}$ is big on every component, so for some n there is a section α of $L_{\overline{X}}^n(-A)$ whose zero locus $\overline{E}_0$ contains Z. For any $r > 0$, setting $\overline{E} = r\overline{E}_0$ we thus have an exact sequence

$$H^0(\overline{X}, L_{\overline{X}}^m(-D)) \to H^0(\overline{E}, L_{\overline{X}}^m(-D)|_{\overline{E}}) \to H^1(\overline{X}, L_{\overline{X}}^{m-nr}(-D + rA)).$$

$L_{\overline{X}}$ is nef, so by Fujita vanishing the rightmost group is zero—and thus the first map is surjective—for some r and any $m \geq nr$. Now apply the previous step and Lemma 6.3. $\square$

Step 4.

Lemma 6.6. *There is a (nonzero) effective divisor E' of X containing Z such that for some integer k and all $m \gg 0$, every section of $L_{\overline{X}}^m(-kD)|_{\overline{E}'}$ whose restriction to Z is a pullback, is the restriction of the pullback of a vanishing section of L_Y^m, where $\overline{E}'$ is the closure of E' in $\overline{X}$.*

Proof. Write $\overline{E} = \overline{E}' + D'$ where D' is supported on the boundary and every component of $\overline{E}'$ meets X. Set $E' = \overline{E}' \cap X$. Note that we have a sequence

$$0 \to \mathcal{O}_{\overline{E}'}(-D') \to \mathcal{O}_{\overline{E}} \to \mathcal{O}_{D'} \to 0$$

and so $\Gamma(\overline{E}', L_{\overline{X}}^m(-kD)|_{\overline{E}'})$ injects in $\Gamma(\overline{E}, L_{\overline{X}}^m(-D)|_{\overline{E}})$ for some fixed k and all $m \geq 0$. Now apply the previous step. $\square$

Step 5. Let $F \subset Y$ be the period image of E'. Applying the induction step to F, it follows that for some n that the map $\Gamma_{van}(F, L_F^n) \to \Gamma(P, L_P^n)$ is surjective. Pulling an appropriate symmetric power of these sections back to $\overline{E}'$ and applying Step 4, we see that these sections extend to vanishing sections of Y, as desired.

To finish the proof, the claim implies L_Y is ample, even for Y non-proper. Indeed, the vanishing sections yield a quasi-finite map $f : Y \to \mathbb{P}^n$, and by Zariski's main theorem this factors as $g \circ i$ for $g : Y' \to \mathbb{P}^n$ finite and $i : Y \to Y'$ an open immersion. As $g^*\mathcal{O}(1)$ is ample, L_Y is too. Now the ring $\bigoplus_n \Gamma_{van}(Y, L_Y^n)$ is integrally closed in $\bigoplus_n \Gamma(Y, L_Y^n)$, since a meromorphic section s which satisfies a monic polynomial relation with coefficients that vanish at the boundary must also vanish at the boundary. Thus, Y' can be projectively embedded with sections vanishing at the boundary. □

7. Applications

We start by making some remarks related to the first two applications below. We may more generally speak of pure polarized integral variations of Hodge structures over a separated Deligne–Mumford stack $\mathcal{M}$ of finite type over $\mathbb{C}$ in the obvious way. For example, for a smooth projective family $\pi : \mathcal{X} \to \mathcal{M}$, the local system $R^k\pi_*\mathbb{Z}$ will underly such a variation. We say that the period map is either quasi-finite or $\mathbb{R}_{\text{an,exp}}$-definable if this is so for the variation pulled back to a finite-type étale atlas.

Recall that the definability condition is again automatic if $\mathcal{M}$ is reduced [3, Theorem 1.3], and is satisfied for all period maps arising from geometry, by Corollary 5.10.

7.1. **Borel algebraicity.** The following is an analog of a theorem proven by Borel [7, Theorem 3.1] (see also [11, Theorem 5.1]) for locally symmetric varieties:

Corollary 7.1. *Let $\mathcal{M}$ be a separated Deligne–Mumford stack of finite type over $\mathbb{C}$ admitting a quasi-finite $\mathbb{R}_{\text{an,exp}}$-definable period map, and let Z be a reduced algebraic space. Then any analytic map $Z^{\text{an}} \to \mathcal{M}^{\text{an}}$ is algebraic.*

Proof. Let $U \to \mathcal{M}$ be a finite-type étale atlas. It is enough to algebraize the base-change of the map $Z^{\text{an}} \to \mathcal{M}^{\text{an}}$ to U along with the descent data, so we may assume $\mathcal{M} = U$. Let Y be the period image of the period map $U \to \Gamma\backslash\Omega$. The composition $Z^{\text{an}} \to U^{\text{an}} \to (\Gamma\backslash\Omega)^{\text{an}}$ is a period map and thus by Corollary 5.1 it follows that $Z^{\text{an}} \to Y^{\text{an}}$ is $\mathbb{R}_{\text{an,exp}}$-definable. As $U \to Y$ is quasi-finite, $Z^{\text{an}} \to U^{\text{an}}$ is also $\mathbb{R}_{\text{an,exp}}$-definable, and therefore by [34, Corollary 4.5] algebraic. □

Applied to a separated Deligne–Mumford moduli stack of smooth polarized varieties with an infinitesimal Torelli theorem, for example, Corollary 7.1 implies that any analytic family of such varieties over (the analytification) of a reduced algebraic base Z is in fact algebraic.

Corollary 7.2. *For $\mathcal{M}$ as above, if $\mathcal{M}$ is in addition reduced, then $\mathcal{M}^{\text{an}}$ admits a unique algebraic structure.*

7.2. **Quasi-projectivity of moduli spaces.** Recall by a well-known result of Keel–Mori [26] that a separated Deligne–Mumford stack $\mathcal{M}$ of finite type over $\mathbb{C}$ admits a coarse moduli space M which is a separated algebraic space of finite type over $\mathbb{C}$.

Corollary 7.3. *Let $\mathcal{M}$ be a separated Deligne–Mumford stack of finite type over $\mathbb{C}$ admitting a quasi-finite $\mathbb{R}_{\text{an,exp}}$-definable period map. Then the coarse moduli space of $\mathcal{M}$ is quasi-projective.*

Proof. The Griffiths bundle exists on the coarse moduli space M as a $\mathbb{Q}$-bundle by general results [28, Lemma 2]. Let $U \to \mathcal{M}$ be a finite-type étale atlas by an algebraic space, so that we have a period map $\varphi : U \to \Gamma\backslash\Omega$. Let Y be the period image. We claim that the map $U \to Y$ factorizes through the coarse moduli space M of $\mathcal{M}$. Let

$\mathcal{M}' \to \mathcal{M}$ and $U' \to U$ be the étale covers corresponding to a normal finite index neat $\Gamma' \subset \Gamma$ with quotient G. Let Y' be the period image of U' in $\Gamma'\backslash\Omega$. Then as the variation on U' is pulled back from Y', the map $U' \to Y'$ factorizes through $\mathcal{M}'$. As $U = [G\backslash U']$ and $\mathcal{M} = [G\backslash \mathcal{M}']$, it follows that $U \to [G\backslash Y']$ factorizes through $\mathcal{M}$. Therefore, the map $U \to [G\backslash Y'] \to Y$ factorizes through M.

Thus we get a quasi-finite map $M \to Y$. By Theorem 6.2, L_Y is ample, so we have an immersion $Y \to \mathbb{P}^n$. We then have a quasi-finite map $M \to \mathbb{P}^n$, which by Zariski's main theorem factors as an open immersion and a finite map. It follows that L_M is ample. □

Remark 7.4. $[G\backslash Y']$ as constructed in the proof deserves to be called the period image of $\mathcal{M}$ in $[\Gamma\backslash\Omega]$, although we have not formally defined these notions.

Corollary 7.3 applies to any (separated finite-type) smooth Deligne–Mumford stack that is the moduli stack of smooth polarized varieties X with an infinitesimal Torelli theorem, for example Calabi–Yau varieties. By work of Viehweg [38], such results are known for varieties X with K_X semi-ample, and so the case of Fano varieties is of particular interest. For concreteness, we deduce some new results about moduli spaces of complete intersections, on which previous work has been done for hypersurfaces by Mumford [31] and more generally by Benoist [4, 5].

We fix a collection of integers $T = (d_1, \cdots, d_c; n)$ with $n \geq 1$, $c \geq 1$ and $2 \leq d_1 \leq \cdots \leq d_c$. Recall that a complete intersection of type T is a closed subscheme of codimension c in $\mathbb{P}^{n+c}_{\mathbb{C}}$ which is the zero locus of c homogeneous polynomials of degrees $d_1, \cdots, d_c$ respectively. Let H be the Zariski-open subset of the Hilbert scheme of $\mathbb{P}^{n+c}_{\mathbb{C}}$ that parametrizes the smooth complete intersections of type T. Let $\mathcal{M}_T$ be the moduli stack of smooth complete intersections polarized by $\mathcal{O}(1)$, i.e. the quotient stack $[PGL^{n+c+1}(\mathbb{C})\backslash H]$.

When $T \neq (2; n)$ Benoist proved that $\mathcal{M}_T$ is a separated smooth Deligne-Mumford stack of finite type [4, Theorem 1.6 and 1.7], and therefore has a coarse moduli space M_T. If in addition $d_1 = \cdots = d_c$ then M_T is an affine scheme, [5, Theorem 1.1.i)], while if $c > 1$ and $d_2 = \cdots = d_c$, M_T is quasi-projective by [5, Corollary 1.2]. Finally, for $T = (3; 2)$, M_T is quasi-projective by [1].

Corollary 7.5. *For all $T \neq (2; n)$, the coarse moduli space M_T is quasi-projective.*

Proof. This follows from Corollary 7.3 and Flenner's infinitesimal Torelli theorem [16, Theorem 3.1], which applies for $T \neq (3; 2)$ and $T \neq (2, 2; n)$ for n even—in particular, to all remaining cases. □

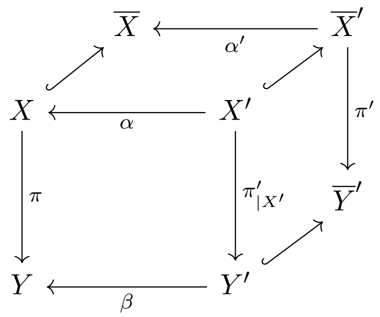
7.3. **A factorization result.**

Theorem 7.6. *Let X be a dense Zariski open subset of a compact Kähler manifold $\overline{X}$, and let $(V_{\mathbb{Z}}, \mathcal{F}^\bullet, Q)$ be a pure polarized integral variation of Hodge structure on X. Assume that the monodromy of $V_{\mathbb{Z}}$ is torsion-free (this is always achieved by going to a finite étale cover of X) and that X is the biggest open subset of $\overline{X}$ on which $V_{\mathbb{Z}}$ extends.*

Then there exist a proper surjective holomorphic map with connected fibres $\pi : X \to Y$ for a normal quasi-projective variety Y such that $(V_{\mathbb{Z}}, \mathcal{F}^\bullet, Q)$ is the pull-back by π of a polarized integral variation of Hodge structure on Y.

Proof. By hypothesis, the monodromy Γ of $(V_{\mathbb{Z}}, \mathcal{F}^\bullet, Q)$ is torsion-free and the associated period map $\varphi : X \to \Gamma\backslash\Omega$ is proper. We denote by $X \xrightarrow{\pi} Y \to \Gamma\backslash\Omega$ its Stein factorization, so that Y is a normal analytic space and $\pi : X \to Y$ is surjective with connected fibres. Since Γ is torsion-free, $(V_{\mathbb{Z}}, \mathcal{F}^\bullet, Q)$ descends to Y. To finish the proof, it remains to prove that Y, a priori only an analytic space, is in fact a quasi-projective variety. We cannot apply directly Theorem 1.1 since X is not assumed to be algebraic. However one can proceed as follows. First observe that thanks to the following result of Sommese Y admits a proper modification $Y' \to Y$ such that Y' is a dense Zariski open subset of compact Kähler manifold $\overline{Y'}$.

Theorem 7.7 (Sommese [37, Proposition III and Remark III-C])**.** *Let X be a dense Zariski open subset in a compact Kähler manifold $\overline{X}$, Y be a complex analytic space and $\pi : X \to Y$ be a surjective proper holomorphic map with connected fibres. Then there exists X' (resp. Y') a dense Zariski open subset in a compact Kähler manifold $\overline{X}'$ (resp. $\overline{Y}'$) and a commutative diagram*



where $\alpha : X' \to X$ (resp. $\beta : Y' \to Y$) are proper modifications and π', $\pi'_{|X'}$ are surjective proper maps with connected fibres.

The composition $Y' \to Y \to \Gamma\backslash\Omega$ endows Y' with a polarized integral variation of Hodge structure. Take $\Gamma' \subset \Gamma$ neat of finite index and let $Y'' \to Y'$ be the base-change along $\Gamma'\backslash\Omega \to \Gamma\backslash\Omega$. If $\overline{Y''}$ denotes a compactification of Y'' whose boundary is a normal crossing divisor, the polarized integral variation of Hodge structure induced on Y'' has unipotent monodromy at infinity. Thanks to Lemma 6.4 the associated Griffiths line bundle $L_{\overline{Y''}}$ is big, hence $\overline{Y''}$ is Moishezon. It follows that the compact Kähler manifold $\overline{Y'}$ is Moishezon, hence it is in fact projective algebraic. Since $Y' \to Y$ is the Stein factorization of the composition $Y' \to Y \to \Gamma\backslash\Omega$, it follows now from Theorem 1.1 and Riemann existence theorem that Y is quasi-projective. □

7.4. **An ampleness criterion for the Hodge bundle.**

Theorem 7.8. *Let $(V_{\mathbb{Z}}, F^\bullet, Q)$ be a pure polarized integral variation of Hodge structure on a (reduced) separated algebraic space X. Assume that the lowest piece of the Hodge filtration F^n is a line bundle. Assume moreover that for any germ of curve $\varphi : \Delta \to X$, the $\mathcal{O}_\Delta$-linear map of $\mathcal{O}_\Delta$-modules $T_\Delta \to \mathrm{Hom}(\varphi^*(F^n), \varphi^*(F^{n-1}/F^n))$ is injective. Then the line bundle F^n is ample on X.*

Observe that for X smooth the last condition is equivalent to asking that the $\mathcal{O}_X$-linear map of $\mathcal{O}_X$-modules $T_X \to \mathrm{Hom}(F^n, F^{n-1}/F^n)$ is injective. Applying the same argument as in section 7.2, we can for example recover a result of Viehweg [38] that the Hodge bundle is ample on the coarse moduli space of a moduli stack of polarized Calabi–Yau varieties.

The proof of Theorem 7.8 is parallel to the proof of Theorem 6.2 (replace Lemma 6.4 by the lemma below whose proof is similar). Note that in fact the latter is a particular case of the former since one easily check that Griffiths line bundle is the lowest piece of the Hodge filtration of the auxiliary variation $\otimes_{p\in\mathbb{Z}} \wedge^{r_p} \mathbb{V}$ where $r_p = \mathrm{rk}\, F^p$.

Lemma 7.9. *Let X be a smooth algebraic variety, $X \subset \overline{X}$ a smooth compactification such that $\overline{X} - X = D$ is a normal crossing divisor. Let $(V_{\mathbb{R}}, F^\bullet, Q)$ be a polarized real* *variation of Hodge structure over X with unipotent monodromies around D and let F^n be Deligne canonical extension of the lowest piece of the Hodge filtration. Assume that F^n is a line bundle. Then F^n is a nef, and moreover it is big if and only if the $\mathcal{O}_X$-linear map of $\mathcal{O}_X$-modules $T_X \to \mathrm{Hom}(F^n, F^{n-1}/F^n)$ is injective.*

Note that the last condition implies that the period map is generically immersive, but the converse is not true.

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